09520 - TEORIA DEI CAMPI (Field Theory)

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Assessment tests

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Contents

0.1 *Thursday 19 February 2019* **1. Field Theory Medley 2. $\pi^0 N$ Elastic Collision**

FIELD THEORY **1.**

1. **Poincaré Invariance of the Klein-Gordon Quantum Field.** The operator valued tempered distribution which describe a Klein-Gordon quantum scalar field is provided by the normal modes expansion

$$\phi(x) = \int \mathrm{D}k \left[a(k)\, e^{-ik\cdot x} + a^\dagger(k)\, e^{ik\cdot x} \right]_{k_0 = \omega_{\mathbf{k}}}$$

where $\int \mathrm{D}k \equiv \int \mathrm{d}\mathbf{k}/(2\pi)^3\, 2\,\omega_{\mathbf{k}}$ while the creation-destruction operator satisfy the canonical commutation relations

$$[\, a(k)\,,\, a(k')\,] = 0 \qquad [\, a(k)\,,\, a^\dagger(k')\,] = (2\pi)^3\, 2k_0\, \delta(\mathbf{k}-\mathbf{k}') \quad (\, k_0 = \omega_{\mathbf{k}}\,)$$

(i) Write the unitary operator of the Poincaré transformations on the Fock space of states of a quantum Klein-Gordon field.

(ii) Evaluate its action upon the creation-destruction operators.

(iii) Verify that the Poincaré transformations look the same for classical and quantum fields.

2. Charge conjugation of a Dirac field is defined to be $\psi^c(x) = \gamma^2\, \psi^*(x)$. Obtain the charge conjugation for the following expressions

$$\overline{\psi} i \partial\!\!\!/ \psi, \overline{\psi}\psi, \overline{\psi}\gamma_5\psi, \overline{\psi}\gamma^\mu\psi, \overline{\psi}\gamma^\mu\gamma_5\psi$$

3. Consider the quantum radiation field in the absence of charged matter. Write the quantum counterpart of the Maxwell equations. Evaluate the commutator $[\,\mathbf{B}(x), \mathbf{B}(y)\,]$ where $\mathbf{B} = \nabla \times \mathbf{A}$.

FIELD THEORY **2.**

The two kinds of Nucleons can be arranged into a doublet of Dirac fields

$$\Psi(x) = \begin{pmatrix} p(x) \\ n(x) \end{pmatrix}$$

transforming according to one of the fundamental representations of SU(2) that is called the Isospin internal symmetry group. The Nucleons interact through the Yukava force carried by a spin-less Iso-scalar field $\pi(x)$, the 1-particle states of which are the neutral π^0 mesons, so that the classical Lagrangian reads

$$\mathcal{L} = \overline{\Psi}(x)\,(i\partial\!\!\!/ - M)\,\Psi(x) + \tfrac{1}{2}\partial_\mu\pi(x)\partial^\mu\pi(x) - \tfrac{1}{2}m^2\pi^2(x) - g\pi(x)\overline{\Psi}(x)\,\Psi(x)$$

where $g > 0$ is the Yukawa coupling while we safely assume[1] $m_p \approx m_n \approx M$.

1. Find all the symmetries of the Action
2. Write the Feynman rules in momentum space
3. Find the lowest order non-polarized differential cross section for the $\pi^0 N$ elastic scattering. Suppose the Nucleon at rest before the collision and the π^0 momenta, before and after the collision, large enough to disregard the meson rest mass.

[1]Experimentally one finds $(m_n - m_p)/(m_n + m_p) \simeq 0.7 \times 10^{-3}$, while $m_{\pi^0}/m_p \simeq 0.14$.

Solution.

FIELD THEORY **1.**

1. It turns out that to each element of the restricted Poincaré group, which is uniquely specified by the ten canonical coordinates

$$\left(\omega^{\mu\nu}, \mathrm{a}^{\lambda}\right) = \left(\boldsymbol{\alpha}, \boldsymbol{\eta}, \mathrm{a}^{\lambda}\right) = (\omega, \mathrm{a})$$

with $|\,\boldsymbol{\alpha}\,| < 2\pi$, $\boldsymbol{\eta} \in \mathbb{R}^3$, $\mathrm{a}^{\lambda} \in \mathbb{R}^4$, there will correspond a unitary operator such that

$$U(\omega, \mathrm{a})\,|\,0\,\rangle = \exp\left\{\frac{i}{\hbar}\,\mathrm{a}^{\mu}\,P_{\mu} - \frac{i}{2\hbar}\,\omega^{\,\rho\sigma}L_{\rho\sigma}\right\}|\,0\,\rangle = |\,0\,\rangle$$

$$\langle\,0\,|\,U^{\,\dagger}(\omega, \mathrm{a}) = \langle\,0\,|\,\exp\left\{-\frac{i}{\hbar}\mathrm{a}^{\mu}\,P_{\mu} + \frac{i}{2\hbar}\,\omega^{\,\rho\sigma}L_{\rho\sigma}\right\} = \langle\,0\,|$$

which means that the vacuum state is Poincaré invariant or, in other words, that $IO(1,3)$ acts trivially on the one dimensional ray of the Fock space generated by the vacuum state for a Klein-Gordon scalar quantum field. In the case of the Klein-Gordon neutral field, the explicit form for the Hermitean generators is provided by the normal ordered expressions

$$\begin{aligned}
P_0 &= \int \mathrm{d}\mathbf{x}\ \tfrac{1}{2} : \Pi^2(x) + \nabla\phi(x)\cdot\nabla\phi(x) + m^2\phi^2(x) : = \sum_{\mathbf{k}}\ \omega_{\mathbf{k}}\, a^{\dagger}_{\mathbf{k}}\, a_{\mathbf{k}} \\
P_k &= \int \mathrm{d}\mathbf{x}\ \ : \Pi(x)\,\partial_k\phi(x) : = \sum_{\mathbf{k}}\ \mathbf{k}\, a^{\dagger}_{\mathbf{k}}\, a_{\mathbf{k}} \\
L_{\imath\jmath} &= \int \mathrm{d}\mathbf{x}\ \ : x_{\imath}\,\Pi(x)\,\partial_{\jmath}\phi(x) - x_{\jmath}\,\Pi(x)\,\partial_{\imath}\phi(x) : \\
&= \sum_{\mathbf{k}}\ \frac{i}{2}\left(k_{\imath}\, a^{\dagger}_{\mathbf{k}}\,\frac{\overleftrightarrow{\partial}}{\partial k^{\jmath}}\, a_{\mathbf{k}}\ -\ k_{\jmath}\, a^{\dagger}_{\mathbf{k}}\,\frac{\overleftrightarrow{\partial}}{\partial k^{\imath}}\, a_{\mathbf{k}}\right) \\
L_{0k} &= x_0\,P_k - m\,X_k(t) = \frac{i}{2}\sum_{\mathbf{p}}\omega_{\mathbf{p}}\ a^{\dagger}_{\mathbf{p}}\,\frac{\overleftrightarrow{\partial}}{\partial\,p^{k}}\,a_{\mathbf{p}} \\
X^k(t) &= \frac{1}{2m}\int \mathrm{d}\mathbf{x}\,x^{\,k} : \Pi^2(x) + \nabla\phi(x)\cdot\nabla\phi(x) + m^2\phi^2(x) :
\end{aligned}$$

Moreover, it can be actually verified that the creation-destruction operators undergo the following changes under a Poincaré transformation: namely,

$$\begin{gathered}
a'(k) \equiv U(\omega, \mathrm{a})\,a(k)\,U^{\,-1}(\omega, \mathrm{a})\ =\ a(k')\,\exp\{-\,i\,k\cdot\mathrm{a}\} \\
k'_{\mu} = \Lambda_{\mu}^{\ \ \nu}\,k_{\nu} \qquad k_0 = \omega_{\mathbf{k}} \qquad g^{\mu\nu}\,k'_{\mu}\,k'_{\nu} = k'^{\,2} = k^2 = m^2 \\
a'^{\dagger}(k)\ \equiv\ U(\omega, \mathrm{a})\,a^{\dagger}(k)\,U^{\,-1}(\omega, \mathrm{a})\ =\ \exp\{\,i\,k\cdot\mathrm{a}\}\,a^{\dagger}(k')
\end{gathered}$$

which endorses the Poincaré invariance, up to a phase factor, of the creation and annihilation operators of the Klein-Gordon quantum scalar field.

As a matter of fact, consider an infinitesimal Poincaré transformation

$$U(\delta\omega,\delta\mathrm{a})\,a(k)\,U^{\,\dagger}(\delta\omega,\delta\mathrm{a}) \simeq a(k) + i\left[\,\delta\mathrm{a}^{\,\mu}\,P_\mu - \tfrac{1}{2}\,\delta\omega^{\,\rho\sigma}L_{\rho\sigma}\,,\,a(k)\,\right]$$

From the canonical commutation relations it is straightforward to show that

$$\begin{aligned} [\,a(k)\,,\,P_\mu\,] &= [\,(\,u_{\,k}\,,\,\phi\,)\,,\,P_\mu\,] = i\,(\,u_{\,k}\,,\,\partial_\mu\,\phi\,) \\ &= \int \mathrm{D}p\;p_\mu\left[\,a(p)\;(\,u_{\,k}\,,\,u_{\,p}) - a^{\,\dagger}(p)\;\left(\,u_{\,k}\,,\,u_{\,p}^{\,*}\right)\,\right] = k_\mu\,a(k) \end{aligned}$$

Moreover we have

$$\begin{aligned} [\,a(k)\,,\,L_{\mu\nu}\,] &= [\,(\,u_{\,k}\,,\,\phi\,)\,,\,L_{\mu\nu}\,] = \int \mathrm{d}\mathbf{x}\;u_{\,k}^{\,*}(t,\mathbf{x})\,i\overleftrightarrow{\partial}_0[\,\phi(t,\mathbf{x})\,,\,L_{\mu\nu}\,] \\ &= -\int \mathrm{d}\mathbf{x}\;u_{\,k}^{\,*}(t,\mathbf{x})\,\overleftrightarrow{\partial}_0\Big(x_\mu\,\partial_\nu\,\phi(t,\mathbf{x}) - x_\nu\,\partial_\mu\,\phi(t,\mathbf{x})\Big) \\ &= \int \mathrm{D}p\;p_\nu\left[\,a^{\dagger}(p)\,\frac{i\partial}{\partial\,p^{\,\mu}}\;(\,u_{\,k}\,,\,u_{\,p}^{\,*}) + a(p)\,\frac{i\partial}{\partial\,p^{\,\mu}}\,(\,u_{\,k}\,,\,u_{\,p})\,\right] - \mu\,\leftrightarrow\,\nu \\ &= -\int \mathrm{D}p\;p_\nu\left[\,a^{\,\dagger}(p)\,\frac{i\partial}{\partial\,k^{\,\mu}}\;(\,u_{\,k}\,,\,u_{\,p}^{\,*}) + a(p)\,\frac{i\partial}{\partial\,k^{\,\mu}}\,(\,u_{\,k}\,,\,u_{\,p})\,\right] - \mu\,\leftrightarrow\,\nu \\ &= -\,\frac{i\partial}{\partial\,k^{\,\mu}}\int \mathrm{D}p\;p_\nu\,a(p)\;(\,u_{\,k}\,,\,u_{\,p}) - \mu\,\leftrightarrow\,\nu \\ &= -\,\frac{i\partial}{\partial\,k^{\,\mu}}\,(k_\nu\,a(k)) - \mu\,\leftrightarrow\,\nu = i\,k_\mu\,\frac{\partial}{\partial\,k^{\,\nu}}\,a(k) - i\,k_\nu\,\frac{\partial}{\partial\,k^{\,\mu}}\,a(k) \end{aligned}$$

where use has been made of the inversion formulæ. Hence, under an infinitesimal Poincaré transformation we get

$$\begin{aligned} &U(\delta\omega,\delta\mathrm{a})\,a(k)\,U^{\,-1}(\delta\omega,\delta\mathrm{a}) \simeq a(k) - i\left[\,a(k)\,,\,\delta\mathrm{a}^{\,\mu}\,P_\mu - \tfrac{1}{2}\,\delta\omega^{\,\rho\sigma}L_{\rho\sigma}\,\right] \\ &= \left\{1 - i\,\delta\mathrm{a}\cdot k - \tfrac{1}{2}\,\delta\omega^{\,\mu\nu}\left(k_\mu\,\frac{\partial}{\partial\,k^{\,\nu}} - k_\nu\,\frac{\partial}{\partial\,k^{\,\mu}}\right)\right\}a(k) \end{aligned}$$

so that we eventually find

$$a^{\,\prime}(k) - a(k)\;\simeq\;\delta a(k)\;\simeq\;\left\{\tfrac{1}{2}\,\epsilon^{\,\mu\nu}\left(k_\nu\,\frac{\partial}{\partial\,k^{\,\mu}} - k_\mu\,\frac{\partial}{\partial\,k^{\,\nu}}\right) - i\,k_\mu\,\epsilon^{\,\mu}\right\}a(k)$$

where we have identified as customary $\delta\mathrm{a}^{\,\mu} \equiv \epsilon^{\,\mu}\,,\;\delta\omega^{\,\mu\nu} \equiv \epsilon^{\,\mu\nu}$. Moreover, the action of an infinitesimal Lorentz transformation on the wave tetra-vector yields

$$a(\Lambda\,k) - a(k)\;\simeq\;a(k+\delta k) - a(k) = \delta k^{\,\mu}\,\frac{\partial}{\partial\,k^{\,\mu}}\,a(k) = \epsilon^{\,\mu\nu}\,k_\nu\,\frac{\partial}{\partial\,k^{\,\mu}}\,a(k)$$

in such a manner that we can finally get the finite transformation rule

$$\begin{aligned} U(\omega,\mathrm{a})\,a(k)\,U^{\,-1}(\omega,\mathrm{a}) &= \exp\{-\,i\,k\cdot\mathrm{a}\}\,a(\Lambda\,k) \\ &\simeq (1 - i\,k_\mu\,\mathrm{a}^{\,\mu} + \cdots)\left(1 - \omega^{\,\mu\nu}\,k_\nu\,\frac{\partial}{\partial\,k^{\,\mu}} + \cdots\right)a(k) \\ &= \left\{1 - i\,k_\mu\,\mathrm{a}^{\,\mu}\,a(k) + \tfrac{1}{2}\,\omega^{\,\mu\nu}\left(k_\nu\,\frac{\partial}{\partial\,k^{\,\mu}} - k_\mu\,\frac{\partial}{\partial\,k^{\,\nu}}\right)\right\}a(k) + \cdots \end{aligned}$$

and consequently

$$a'(k) \;\equiv\; U(\omega,\mathrm{a})\,a(k)\,U^{\,\dagger}(\omega,\mathrm{a}) \;=\; a(\Lambda\,k)\exp\{-\,i\,k\cdot\mathrm{a}\} \qquad k_0=\omega_{\mathbf{k}}$$

as claimed. It is worthwhile to notice that by repeating the very same steps for the inverse Poincaré unitary similarity transformation we obtain

$$a'(k) \;\equiv\; U^{\,-1}(\omega,\mathrm{a})\,a(k)\,U(\omega,\mathrm{a}) \;=\; a(\Lambda^{-1}k)\exp\{\,i\,k\cdot\mathrm{a}\} \qquad (\,k_0=\omega_{\mathbf{k}}\,)$$

with

$$U^{\,-1}(\omega,\mathrm{a}) = U(-\,\omega,-\,\mathrm{a}) = U^{\,\dagger}(\omega,\mathrm{a})$$

which implies in turn

$$\begin{aligned} a'(k'\,) \;&\equiv\; U^{\,\dagger}(\omega,\mathrm{a})\,a(\Lambda\,k)\,U(\omega,\mathrm{a}) \\ &=\; a(k)\exp\{\,i\,k\cdot\mathrm{a}\} \qquad\qquad (\,k_0=\omega_{\mathbf{k}}\,) \end{aligned}$$

showing that for any homogeneous Lorentz transformation the creation and annihilation operators are invariant. From the transformation law of the creation-destruction operators we immediately obtain the following identity between operator valued tempered distributions, which looks the very same as that one for the classical scalar field, *viz.,*

$$\begin{aligned} \phi'(x'\,) \;&\equiv\; U^{\,-1}(\omega,\mathrm{a})\,\phi(x'\,)\,U(\omega,\mathrm{a}) \\ &=\; \int \mathrm{D}k'\,\Big[\,a'(k'\,)\,\exp\{-i\,k'\cdot x'\} + \mathrm{h.\,c.}\Big]_{k_0'=\omega_{\mathbf{k}'}} \\ &=\; \int \mathrm{D}(\Lambda k)\,\Big[\,e^{\,ik\cdot\mathrm{a}}\,a(k)\,e^{\,-\,ik\cdot(x+\mathrm{a})} + \mathrm{h.\,c.}\Big]_{k_0=\omega_{\mathbf{k}}} \;=\; \phi(x) \end{aligned}$$

Concerning the discrete symmetry transforms, parity and time reversal, we find instead

$$\begin{aligned} \phi'(x') \;&=\; \mathcal{P}\phi(x')\mathcal{P}^{\,-1} = \mathcal{P}\phi(t,-\,\mathbf{x})\mathcal{P}^{\,\dagger} \\ &=\; \int \mathrm{d}\mathbf{k}\;a_{\,-\,\mathbf{k}}\,u_{\mathbf{k}}(t,-\,\mathbf{x}) \;+\; \mathrm{H.c.} \\ &=\; \int \mathrm{d}\mathbf{k}\;a_{\,-\,\mathbf{k}}\,[\,(2\pi)^3\,2\omega_{\mathbf{k}}\,]^{-\,\frac{1}{2}}\exp\{-\,it\omega_{\mathbf{k}} - i\,\mathbf{k}\cdot\mathbf{x}\,\} \;+\; \mathrm{H.c.} \\ &=\; \int \mathrm{d}\mathbf{k}\;a_{\mathbf{k}}\,[\,(2\pi)^3\,2\omega_{\mathbf{k}}\,]^{-\,\frac{1}{2}}\exp\{-\,it\omega_{\mathbf{k}} + i\,\mathbf{k}\cdot\mathbf{x}\,\} \;+\; \mathrm{H.c.} \\ &=\; \phi(x) \end{aligned}$$

with $\mathcal{P}=\mathcal{P}^{\,-1}=\mathcal{P}^{\dagger}$ ($\mathcal{P}^{\,2}=\mathbb{I}$) unitary and self-adjoint operator on the Fock space, while

$$\phi'(x') \;=\; \mathcal{T}\,\phi(x')\,\mathcal{T}^{\,-1} = \mathcal{T}\,\phi(-\,t,\mathbf{x})\,\mathcal{T}^{\,\dagger}$$

$$
\begin{aligned}
&= \int \mathrm{d}\mathbf{k}\; a_{-\mathbf{k}}\, u^*_{\mathbf{k}}(-t,\mathbf{x}) \;+\; \text{H.c.} \\
&= \int \mathrm{d}\mathbf{k}\; a_{-\mathbf{k}}\,[\,(2\pi)^3\, 2\omega_{\mathbf{k}}\,]^{-\frac{1}{2}} \exp\{-\,it\omega_{\mathbf{k}} - i\,\mathbf{k}\cdot\mathbf{x}\,\} \;+\; \text{H.c.} \\
&= \int \mathrm{d}\mathbf{k}\; a_{\mathbf{k}}\,[\,(2\pi)^3\, 2\omega_{\mathbf{k}}\,]^{-\frac{1}{2}} \exp\{-\,it\omega_{\mathbf{k}} + i\,\mathbf{k}\cdot\mathbf{x}\,\} \;+\; \text{H.c.} \\
&= \phi(x)
\end{aligned}
$$

where use has been made of the anti-linear and anti-unitary property of the time reversal operator $\forall\, \alpha, \beta \in \mathbb{C} \vee |\, u\,\rangle\,, |\, v\,\rangle \in \mathcal{F}$

$$
\mathcal{T}\,(\alpha\,|\,u\,\rangle + \beta\,|\,v\,\rangle) = \alpha^*\,\mathcal{T}\,|\,u\,\rangle + \beta^*\,\mathcal{T}\,|\,v\,\rangle
$$

$$
\langle\, u\,|\,\mathcal{T}\,\mathcal{T}^\dagger\,|\,v\,\rangle = \langle\, v\,|\,u\,\rangle
$$

and of the definitive transformation rules

$$
\mathcal{T}\,\mathbf{P}\,\mathcal{T}^{-1} = -\,\mathbf{P} \qquad \mathcal{T}\,L_{\mu\nu}\,\mathcal{T}^{-1} = -\,L_{\mu\nu}
$$

$$
\mathcal{T}\,a_{\mathbf{k}}\,\mathcal{T}^{-1} = a_{-\mathbf{k}} \qquad [\,\mathcal{T}\,,\,P_0\,] = 0
$$

2. To be definite, consider Graßmann valued Dirac spinor fields, which obey the complex conjugation rule $(\,\psi_1\psi_2\,)^* = \psi_2^*\psi_1^* = -\,\psi_1^*\psi_2^*$. Thus, in the chiral representation of the Clifford algebra - with γ^0 and γ_5 real matrices - we obtain

$$
\begin{aligned}
\overline{\psi^{\,c}}\, i\partial\!\!\!/\, \psi^{\,c} &= \psi^\top\gamma^0\gamma^2\,\gamma^\mu\,\gamma^2\, i\partial_\mu\psi^* = \psi^\top\gamma^0\,\gamma^{2*}\, i\partial_\mu\psi^* \\
&= \left(\,\overline{\psi}\, i\partial\!\!\!/\psi\,\right)^* = \overline{\psi}\, i\partial\!\!\!/\psi \\
\overline{\psi^{\,c}}\,\psi^{\,c} &= -\,\psi^\top\gamma^2\,\gamma^0\,\gamma^2\,\psi^* = -\,\psi^\top\gamma^0\,\psi^* = (\,\overline{\psi}\,\psi\,)^* = \overline{\psi}\,\psi \\
\overline{\psi^{\,c}}\,\gamma_5\,\psi^{\,c} &= -\,\psi^\top\gamma^2\,\gamma^0\,\gamma_5\,\gamma^2\,\psi^* \\
&= \psi^\top\gamma^0\,\gamma_5\,\psi^* = -(\,\overline{\psi}\,\gamma_5\,\psi\,)^* = -\,\overline{\psi}\,\gamma_5\,\psi \\
\overline{\psi^{\,c}}\,\gamma^\mu\,\psi^{\,c} &= \psi^\top\gamma^0\gamma^2\,\gamma^\mu\,\gamma^2\,\psi^* = \psi^\top\gamma^0\,\gamma^{2*}\,\psi^* \\
&= -\left(\,\overline{\psi}\,\gamma^\mu\,\psi\,\right)^* = -\,\overline{\psi}\,\gamma^\mu\,\psi \\
\overline{\psi^{\,c}}\,\gamma^\mu\,\gamma_5\,\psi^{\,c} &= -\,\psi^\top\gamma^0\gamma^2\,\gamma^\mu\,\gamma^2\,\gamma_5\,\psi^* = -\,\psi^\top\gamma^0\,\gamma^{2*}\,\gamma_5\,\psi^* \\
&= \left(\,\overline{\psi}\,\gamma^\mu\,\gamma_5\,\psi\,\right)^* = \overline{\psi}\,\gamma^\mu\,\gamma_5\,\psi
\end{aligned}
$$

3. In order to solve Maxwell equations in terms of the gauge potential it is necessary to introduce the so called subsidiary or auxiliary condition. If Lorentz covariance has to be manifestly maintained, then it is convenient to

select the simplest choice: namely, the so called Feynman gauge. Hence we start from the classical Lagrangian for the radiation field

$$\mathcal{L} = -\,\tfrac{1}{4}\,F_{\mu\nu}\,F^{\,\mu\nu} + \tfrac{1}{2}\,B^{\,2} + A^{\mu}\partial_{\mu}B$$

where

$$F_{0k} = E^{\,k} = -\,\partial_0 A^k -\,\nabla_k A^0 \qquad\qquad F_{rs} = \varepsilon_{srk}\,B^{\,k} \qquad \varepsilon_{123} = 1$$

while $B(x)$ is the auxiliary scalar field, so that the Euler-Lagrange equations read

$$\begin{aligned} \partial^{\,\mu}F_{\mu\nu} + \partial_{\nu}B &= 0 \\ \partial\cdot A &= B \end{aligned}$$

and can be recast in the simplest form

$$\Box A^{\mu}(x) = 0 \qquad\qquad \partial\cdot A(x) = B(x)$$

that imply in turn $\Box B(x) = 0$. Now, in the classical case the field equations can be further simplified, without loss of generality, by setting $B(x) \equiv 0$, that actually corresponds to the Lorenz condition. In so doing we obtain the most general solution in the form of a normal modes expansion

$$A^{\mu}(x) = \int \mathrm{D}k \sum_{A=1,2,L} g_A(k)\,\varepsilon^{\,\mu}_A(k)\,e^{-i\,k\cdot x} \;+\; \text{c.c.} \qquad (\,k_0 = \mathrm{k} = |\,\mathbf{k}\,|\,)$$

with $\mathrm{D}k = \mathrm{d}\mathbf{k}/(2\pi)^3 2\mathrm{k}$ and $g_A(k)$ arbitrary complex coefficients. The three linear and real polarization vectors are defined to be

$$\varepsilon^{\,\mu}_A(k) = \left\{ \begin{array}{lll} (0,\boldsymbol{\varepsilon}_A(\mathbf{k}) & \boldsymbol{\varepsilon}_A(\mathbf{k})\cdot\mathbf{k} = 0 & \text{for} \quad A = 1,2 \\ (\,1,\mathbf{k}/\mathrm{k}\,) & \quad k_0^2 = \mathrm{k}^2 & \text{for} \quad A = L \end{array} \right.$$

in such a manner that the Lorenz condition holds always true and $B = 0$. Notice that in such a circumstance we recover the second pair of the Maxwell equations $\partial^{\,\mu}F_{\mu\nu} = 0$ for the radiation field. In the quantum case it is utmost convenient to keep the Feynman gauge, which endorses locality of the gauge potential operator, so that we find

$$A^{\mu}(x) = \int \mathrm{D}k \sum_{A=1,2,L,S} g_A(k)\,\varepsilon^{\,\mu}_A(k)\,e^{-\,i\,k\cdot x} \;+\; \text{H.c.} \qquad (\,k_0 = \mathrm{k}\,)$$

where the scalar polarization vector $\varepsilon^{\,\mu}_S = \frac{1}{2}(\,1,-\,\mathbf{k}/\mathrm{k}\,)$ has been employed so that

$$i\partial\cdot A(x) = iB(x) = \int \mathrm{D}k\;\mathrm{k}\,g_S(k)\,e^{-\,i\,k\cdot x} -\; \text{H.c.} \qquad (\,k_0 = \mathrm{k}\,)$$

In the quantum case the complex coefficients of the classical normal modes expansion will turn into creation and destruction operators which fulfill the canonical commutation relations

$$[\,g_A(k)\,,\,g_{A'}(k')\,] = 0 \qquad\qquad [\,g_A(k)\,,\,g^{\dagger}_{A'}(k')\,] = (2\pi)^3 2\mathrm{k}\,\delta(\mathbf{k}-\mathbf{k}')\,\eta_{AA'}$$

with

$$\eta_{AB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad\qquad (\,A, B = 1, 2, L, S\,)$$

The second pair of the Maxwell equations is recovered only in the physical subspace $\mathcal{H}_{\mathrm{phys}} \subset \mathcal{F}$ of the Fock space, which is selected by the auxiliary condition

$$B^{(-)}(x)|\mathrm{phys}\rangle = 0 \quad \Longleftrightarrow \quad g_S(k)|\mathrm{phys}\rangle = 0 \quad \forall\,\mathbf{k} \in \mathbb{R}^3$$

that yields

$$\langle \mathrm{phys}|\,\partial^{\mu}F_{\mu\nu} + \partial_{\nu}B\,|\mathrm{phys}'\rangle = \langle \mathrm{phys}|\,\partial^{\mu}F_{\mu\nu}\,|\mathrm{phys}'\rangle = 0$$

From the canonical commutation relations for the gauge potential in the Feynman gauge, *viz.*,

$$[\,A^{\mu}(x)\,,\,A^{\nu}(0)\,] = i\hbar\,g^{\mu\nu}\,D_0(x)$$

where $D_0(x) = \lim_{m\to 0} D(x;m)$ is the mass-less Pauli-Jordan distribution, we readily obtain the gauge and translation invariant commutation relations

$$\begin{aligned} [\,F_{\mu\nu}(x)\,,\,F_{\rho\sigma}(0)\,] &= (-i\hbar)\{g_{\nu\sigma}\,\partial_{\mu}\partial_{\rho} - g_{\mu\sigma}\,\partial_{\nu}\partial_{\rho} \\ &- g_{\nu\rho}\,\partial_{\mu}\partial_{\sigma} + g_{\mu\rho}\,\partial_{\nu}\partial_{\sigma}\}D_0(x) \end{aligned}$$

whence

$$[\,B_x(\mathrm{x})\,, B_x(0)\,] = i\hbar\left(\triangle - \partial_x^2\right)D_0(\mathrm{x})$$
$$[\,B_x(\mathrm{x})\,, B_y(0)\,] = -\,i\hbar\,\partial_x\partial_y\,D_0(\mathrm{x})$$

et cetera, where $\mathrm{x} = (t, x, y, z)$ and $(B_x, B_y, B_z) = (F_{32}, F_{13}, F_{21})$ as usual.

FIELD THEORY **2.**

The Lagrangian is invariant under the full Lorentz group and the discrete charge conjugation symmetry, under the internal SU(2) Isospin transforms on the spinor fields

$$\Psi(x) \quad \longmapsto \quad \Psi'(x) = \exp\left\{\tfrac{1}{2}\,i\sigma_a\,\theta_a\right\}\Psi(x) \qquad\qquad (\,a = 1, 2, 3\,)$$

as well as the overall phase transformation on the SU(2) spinor doublet

$$\Psi(x) \quad \longmapsto \quad \Psi'(x) = \mathrm{e}^{\,i\varphi}\,\Psi(x)$$

where σ_a are the Pauli matrices while

$$0 \le \theta < 2\pi \qquad 0 \le \varphi < 2\pi \qquad \theta = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2}$$

are the canonical coordinates of the internal symmetry group SU(2)×U(1). The invariance under the Abelian group of the phase transformations leads to conservation of the barion number B. Thus, if we measure the charge Q in units of the proton charge e, then we can write the relation $Q = T_3 + \frac{1}{2}B$ where

$$\begin{aligned} T_3 &= \int \mathrm{d}\mathbf{x}\,\Psi^{\,\dagger}(t,\mathbf{x})\,\tfrac{1}{2}\,\sigma_3\Psi(t,\mathbf{x}) \\ &= \tfrac{1}{2}\int \mathrm{d}\mathbf{x}\,[\,p^{\dagger}(t,\mathbf{x})\,p(t,\mathbf{x}) - n^{\dagger}(t,\mathbf{x})\,n(t,\mathbf{x})\,] \\ B &= \int \mathrm{d}\mathbf{x}\,[\,p^{\dagger}(t,\mathbf{x})\,p(t,\mathbf{x}) + n^{\dagger}(t,\mathbf{x})\,n(t,\mathbf{x})\,] \\ \frac{Q}{e} &= \int \mathrm{d}\mathbf{x}\;p^{\dagger}(t,\mathbf{x})\,p(t,\mathbf{x}) \end{aligned}$$

The momentum space Feynman rules are the very same for both kinds of Nucleons as well as the Feynman rules for the incoming and outgoing particles and antiparticles: namely

- pion propagator: $D_F(k) = i\,[\,k^2 - m^2 + i\varepsilon\,]^{-1}$
- spinor propagator: $S^F_{\alpha\beta}(p) = i(\not p + M)_{\alpha\beta}\,(p^2 - M^2 + i\varepsilon)^{-1}$
- pion-Nucleon-Nucleon vertex: $-\,ig \qquad (\,p_1 + k - p_2 = 0\,)$
- for each loop of internal line labeled by ℓ : $\int \mathrm{d}^4\ell/(2\pi)^4$
- a factor $(-\,1)$ for each fermion loop
- incoming Nucleon: $u_r(p)$
- outgoing Nucleon: $\bar u_{r'}(p')$
- incoming anti-Nucleon: $\bar v_r(p)$
- outgoing anti-Nucleon: $v_{r'}(p')$

The above Feynman rules give at once the lowest order $O(g^2)$ amplitude for the $\pi^0 N$ elastic scattering: namely,

$$
\begin{aligned}
i\,\mathcal{M}_{rr'}(\mathbf{k},\mathbf{k}') &= \bar{u}_{r'}(p')\,(-ig)\,S(p+k)\,(-ig)\,u_r(p) + \{k \leftrightarrow -k'\} \\
&= -ig^2\,\bar{u}_{r'}(p')\left[\frac{\not p + \not k + M}{(p+k)^2 - M^2} + \frac{\not p - \not k' + M}{(p-k')^2 - M^2}\right] u_r(p) \\
&= -ig^2\,\bar{u}_{r'}(p')\left[\frac{\not k + 2M}{(p+k)^2 - M^2} - \frac{\not k' - 2M}{(p-k')^2 - M^2}\right] u_r(p)
\end{aligned}
$$

where

$$
p + k - k' = p' \qquad k^2 = k'^2 = m^2 \qquad p^2 = p'^2 = M^2
$$

whereas use has been made of the spin-states equation $(\not p - M)\,u_r(p) = 0$. Moreover, if we select the incoming Nucleon rest frame $\mathbf{p} = 0$ we find

$$
(p+k)^2 - M^2 = m^2 + 2M\omega \qquad (p-k')^2 - M^2 = m^2 - 2M\omega'
$$

where $k^\mu = (\omega, \mathbf{k})$ with $\omega \approx |\mathbf{k}|$, whereas $k'_\mu = (\omega', -\mathbf{k}')$ with $\omega' \approx |\mathbf{k}'|$, in such a manner that we can write

$$
\begin{aligned}
\mathcal{M}_{rr'}(\mathbf{k},\mathbf{k}') &= ig^2\,\bar{u}_{r'}(p')\left[\frac{\not k + 2M}{2M\omega + m^2} + \frac{\not k' - 2M}{2M\omega' - m^2}\right] u_r(p) \\
&\approx \frac{ig^2}{2M\omega\omega'}\,\bar{u}_{r'}(p')\,\left[2M(\omega' - \omega) + \not k\omega' + \not k'\omega\right] u_r(p)
\end{aligned}
$$

for ultra-relativistic incident and scattered pions. Then we can approximate

$$
\begin{aligned}
&\mathcal{M}^*_{rr'}(\mathbf{k},\mathbf{k}') \\
\approx\ &\frac{-ig^2}{2M\omega\omega'}\,\bar{u}_r(p)\left[2M(\omega' - \omega) + \not k\omega' + \not k'\omega\right] u_{r'}(p')
\end{aligned}
$$

and if we set

$$
Q \equiv 2M\Delta\omega + \not k\omega' + \not k'\omega \qquad (\Delta\omega \equiv \omega' - \omega)
$$

then we can definitely write

$$
\begin{aligned}
\left\langle |\mathcal{M}(\mathbf{k},\mathbf{k}')|^2 \right\rangle &= \tfrac{1}{2}\sum_{r=1,2}\sum_{r'=1,2} |\mathcal{M}^*_{rr'}(\mathbf{k},\mathbf{k}')|^2 \\
&\approx \frac{g^4}{8M^2\omega^2\omega'^2}\,\mathrm{tr}\left[(\not p' + M)\,Q\,(\not p + M)\,Q\right] \\
&= \frac{g^4}{8M^2\omega^2\omega'^2}\,\mathrm{tr}\left[(\not p + M + \not k - \not k')\,Q\,(\not p + M)\,Q\right] \\
&= \frac{g^4}{8M^2\omega^2\omega'^2}\,\mathrm{tr}\left[\mathbb{A}_1 + \mathbb{A}_2 + \mathbb{A}_3\right]
\end{aligned}
$$

with

$$\mathbb{A}_1 = (\not{p} + M)\, Q\, (\not{p} + M)\, Q \qquad \mathbb{A}_2 = (\not{k} - \not{k}')\, Q\, \not{p}\, Q \qquad \mathbb{A}_3 = M(\not{k} - \not{k}')\, Q^2$$

Explicit trace calculations yields

$$\begin{aligned}
\operatorname{tr}\mathbb{A}_1 &= \operatorname{tr}[\,(\not{p} + M)\, Q\, (\not{p} + M)\, Q\,] \\
&= \operatorname{tr}[\,\not{p}\, Q\, \not{p}\, Q\,] + 2M \operatorname{tr}[\,Q\, \not{p}\, Q\,] + M^2 \operatorname{tr} Q^2 \\
&= 16M^4 \Delta\omega^2 + \operatorname{tr}[\,\not{p}(\not{k}\omega' + \not{k}'\omega)\not{p}(\not{k}\omega' + \not{k}'\omega)\,] \\
\operatorname{tr}[\,\not{p}\, Q\, \not{p}\, Q\,] &\approx 16M^4 \Delta\omega^2 + 8\,(\,p \cdot k\,)^2\,\omega'^2 + 8\,(\,p \cdot k'\,)^2\,\omega^2 \\
&- 8M^2\,\omega\,\omega'\,(k \cdot k'\,) + 16\,\omega\,\omega'\,(\,p \cdot k\,)(\,p \cdot k'\,) \\
&= 16M^4 \Delta\omega^2 + 32M^2\omega^2\,\omega'^2 - 8M^2\,\omega\,\omega'\,(k \cdot k'\,) \\
M^2 \operatorname{tr} Q^2 &\approx 16M^4 \Delta\omega^2 + 8M^2\,\omega\,\omega'(\,k \cdot k'\,) \\
2M \operatorname{tr}[\,Q\, \not{p}\, Q\,] &= 8M^2\,\Delta\omega \operatorname{tr}[\,\not{p}(\not{k}\omega' + \not{k}'\omega)\,] \approx 64M^3\,\omega\,\omega'\,\Delta\omega
\end{aligned}$$

Then we definitely find

$$\begin{aligned}
\operatorname{tr}\mathbb{A}_1 &\approx 32M^2\,(\omega\,\omega' + M\,\Delta\omega)^2 \\
\operatorname{tr}\mathbb{A}_2 &= \operatorname{tr}[\,(\not{k} - \not{k}')\, Q\, \not{p}\, Q\,] \;=\; 4M^2\,\Delta\omega^2 \operatorname{tr}[\,(\not{k} - \not{k}')\,\not{p}\,] \\
&+ \operatorname{tr}[\,(\not{k} - \not{k}')\,(\,\not{k}\,\omega' + \not{k}'\,\omega\,)\,\not{p}\,(\,\not{k}\,\omega' + \not{k}'\,\omega\,)\,] \\
&\approx -\,16\,M^3\,\Delta\omega^3 - 8M\omega\omega'\Delta\omega\,(\,k \cdot k'\,) \\
\operatorname{tr}\mathbb{A}_3 &= 4M^2\Delta\omega\,\operatorname{tr}[\,(\,\not{k} - \not{k}'\,)(\,\not{k}\omega' + \not{k}'\omega\,)\,] \\
&\approx -\,16M^2\Delta\omega^2\,(\,k \cdot k'\,)
\end{aligned}$$

where we have taken into account the kinematics of the initial Nucleon rest frame $\mathbf{p} = 0$ that yields

$$p \cdot k = M\omega \qquad p \cdot k' = M\omega' \qquad k \cdot k' \approx 2\omega\omega' \sin^2\frac{\theta}{2}$$

Putting altogether we eventually get

$$\begin{aligned}
\left\langle\,|\,\mathcal{M}(\mathbf{k},\mathbf{k}'\,)\,|^2\right\rangle &= 2g^4 \operatorname{tr}[\,\mathbb{A}_1 + \mathbb{A}_2 + \mathbb{A}_3\,]\,(\,4M\omega\omega'\,)^{-2} \\
&\approx 2g^4\left[\,2\left(1 + \frac{M}{\omega} - \frac{M}{\omega'}\right)^2 - M\Delta\omega\left(\frac{1}{\omega} - \frac{1}{\omega'}\right)^2\right. \\
&- \left.\frac{\Delta\omega}{M}\,\sin^2\frac{\theta}{2} + 2\left(2 - \frac{\omega}{\omega'} - \frac{\omega'}{\omega}\right)\sin^2\frac{\theta}{2}\,\right]
\end{aligned}$$

Let us close with the calculation of the incident flux factor and the final phase space volume in the massless pion limit: in this limit one immediately recovers the corresponding quantities of the Compton effect, *viz.*

$$
\begin{aligned}
\mathrm{d}\sigma &= \tfrac{1}{4}\,(\,p\cdot k\,)^{-1}\;\cdot\;\tfrac{1}{2}\sum_{r,r\,'=1,2}\;|\,\mathcal{M}_{\,rr\,'}(\mathbf{k},\mathbf{k}'\,)\,|^{2}\\
&\times\;\int\frac{\mathrm{d}\mathbf{k}'}{(2\pi)^3\,2\omega'}\int\frac{\mathrm{d}\mathbf{p}'}{(2\pi)^3\,2E'}\;(2\pi)^4\;\delta^{(4)}(k+p-k'-p'\,)\\
&=\;(\,4M\omega\,)^{-1}\;\left\langle\,|\,\mathcal{M}(\mathbf{k},\mathbf{k}'\,)\,|^2\right\rangle\\
&\times\;\int_0^\infty\frac{\omega'\,\mathrm{d}\omega'}{(2\pi)^3\,2}\int\frac{\mathrm{d}\Omega}{2E'(\omega'\,)}\;(2\pi)\,\delta\,(\omega'+E'(\omega'\,)-M-\omega)
\end{aligned}
$$

in which

$$E'(\omega'\,)\equiv\sqrt{\omega'^{\,2}+\omega^{\,2}-2\omega\omega'\,\cos\theta+M^{\,2}}$$

$$\mathrm{d}\Omega=\mathrm{d}\phi\,(-\,\mathrm{d}\cos\theta)\qquad\qquad(\,0\le\theta\le\pi\,,\;0\le\phi\le2\pi\,)$$

From the theory of the tempered distributions we get the well known relation

$$
\begin{aligned}
&\int_0^\infty\frac{\omega'\,\mathrm{d}\omega'}{E'(\omega'\,)}\;\delta\,(\omega'+E'(\omega'\,)-M-\omega)\;f(\omega'\,)\\
=\;&\left[\frac{\omega'\,f(\omega'\,)}{|\,E'(\omega'\,)+\,\omega'-\omega\cos\theta\,|}\right]_{\omega'=\widetilde{\omega}'}\qquad\qquad[\,\forall\,f\in\mathcal{S}(\mathbb{R})\,]
\end{aligned}
$$

where

$$\widetilde{\omega}'+E'(\widetilde{\omega}'\,)=\omega+M\;\Leftrightarrow\;\widetilde{\omega}'\;\equiv\;\frac{\omega\,M}{M+\omega\,(1-\cos\theta)}$$

in such a manner that

$$\left[\frac{\omega'}{|\,E'(\omega'\,)+\,\omega'-\omega\cos\theta\,|}\right]_{\omega'=\widetilde{\omega}'}=\frac{\omega'}{M+\omega\,(1-\cos\theta)}=\frac{\widetilde{\omega}'^{\,2}}{\omega\,M}$$

so that we come to the differential cross-section in the Compton laboratory frame $\mathbf{p}=0$: namely,

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)=\frac{1}{64\pi^2}\cdot\frac{\omega'}{\omega}\cdot\frac{\langle\,|\,\mathcal{M}(\mathbf{k},\mathbf{k}'\,)\,|^2\rangle}{M^{\,2}+2M\omega\sin^{\,2}(\theta/2)}$$