2nd Semester Course

Intermediate relativistic quantum field theory
(a next-to-basic course for primary education)

Roberto Soldati

April 27, 2009

Contents

Chapter 1

Feynman rules

We are now in order to develop perturbation theory, which will provide the fundamental tool to calculate the probability amplitudes for all the physical processes involving relativistic quantized fields in mutual interaction. This is precisely the ambitious final task of the quantum theory of relativistic wave fields. In this aim, I will consider the paradigmatic and simplest cases of the self-interacting real scalar field, together with the Dirac field interacting *á la* Yukawa, the generalizations to any other set of mutually interacting fields of any mass, spin and charges being admittedly straightforward.

1.1 Connected Green's functions

Let me start from the self-interacting real scalar field theory. We recall the classical action for the real scalar relativistic wave field, that is

$$S\,[\,\phi\,] = S_0\,[\,\phi\,] - V\,[\,\phi\,]$$

$$S_0\,[\,\phi\,] = \int \mathrm{d}x\ \tfrac{1}{2}\,\{g^{\,\mu\nu}\,\partial_\mu\phi(x)\partial_\nu\phi(x) - m^2\,\phi^{\,2}(x)\}$$

$$V\,[\,\phi\,] = \frac{\lambda}{4!}\int \mathrm{d}x\ \phi^{\,4}(x)$$

The generating functional for the self-interacting real scalar field theory is defined by

$$\begin{aligned} Z\,[\,J\,] \;&=\; \left\langle T\,\exp\left\{ \mathrm{i}\int \mathrm{d}x\ \phi(x)\,J(x)\right\}\right\rangle_0 \\ &\stackrel{\mathrm{def}}{=}\; \sum_{n=0}^{\infty}\frac{\mathrm{i}^{\,n}}{n!}\int \mathrm{d}x_1\,J(x_1)\cdots\int \mathrm{d}x_n\,J(x_n) \\ &\times\; \langle 0\,|\,T\,\phi\,(x_1)\cdots\phi\,(x_n)\,|\,0\,\rangle \end{aligned} \tag{1.1}$$

where $J(x)$ are the classical external sources with the canonical dimensions $[\,J\,] = \mathrm{eV}^3$. The vacuum expectation values of the chronological ordered products of n scalar field operators at different spacetime points are named the $n-$point Green functions of the field theory. By construction, the latter ones can be expressed as functional derivatives of the generating functional

$$\begin{aligned} G^{(n)}(x_1,\cdots,x_n) &\equiv \langle 0\,|\,T\,\phi\,(x_1)\cdots\phi\,(x_n)\,|\,0\rangle \\ &= (-\,\mathrm{i}\,)^n\,\delta^{(n)}\,Z\,[\,J\,]\,/\,\delta\,J(x_1)\cdots\delta\,J(x_n)\,\rfloor_{J=0} \end{aligned} \tag{1.2}$$

We remind that taking one functional derivative of the generating functional (1.1) we find

$$\frac{\delta\,Z\,[\,J\,]}{\mathrm{i}\,\delta\,J(x)} = \left\langle T\,\phi(x)\,\exp\left\{\mathrm{i}\int dy\;\phi(y)\,J(y)\right\}\right\rangle_0 \tag{1.3}$$

The generating functional for the free field theory corresponding to $\lambda = 0$ has been explicitly computed in the first part of these notes. Moreover, a functional integral representation has been obtained after transition to the euclidean formulation and use of the Zeta function regularization : namely,

$$\begin{aligned} Z_0\,[\,J\,] &= \exp\left\{-\tfrac{1}{2}\int \mathrm{d}x \int \mathrm{d}y\;\, J(x)\,D_F\,(x-y)\,J(y)\right\} \\ &\overset{\mathrm{def}}{=} \mathcal{N}\int \mathfrak{D}\phi\;\exp\left\{\mathrm{i}\,S_0\,[\,\phi\,] + \mathrm{i}\int \mathrm{d}x\;\phi(x)\,J(x)\right\} \\ S_0\,[\,\phi\,] &= -\,\tfrac{1}{2}\int \mathrm{d}x\;\phi(x)\,(\,\Box + m^2 - \mathrm{i}\varepsilon\,)\;\phi(x) \\ \mathcal{N} &= \text{constant} \times \Big(\det\,\|\,(\Box + m^2)/\mu^2\,\|\Big)^{1/2} \\ &\overset{\mathrm{def}}{=} \exp\left\{(\,\text{Volume}\,)\,\frac{\mathrm{i}\,m^4}{32\pi^2}\left(\ln\,\frac{m}{\mu} - \frac{3}{4}\right)\right\} \qquad \text{(Zeta regularization)} \\ Z_0\,[\,0\,] &= \mathcal{N}\int \mathfrak{D}\phi\;\exp\{\,\mathrm{i}\,S_0\,[\,\phi\,]\,\} = 1 \end{aligned}$$

It is immediate to gather that

$$V\,[\,\delta\,/\,\mathrm{i}\,\delta\,J\,]\,Z_0\,[\,J\,] = \mathcal{N}\int \mathfrak{D}\phi\;V\,[\phi]\,\exp\left\{\mathrm{i}\,S_0\,[\,\phi\,] + \mathrm{i}\int \mathrm{d}x\;\phi(x)\,J(x)\right\}$$

in such a manner that I can formally define the generating functional for the real self-interacting scalar field theory as follows

$$\begin{aligned} Z\,[\,J\,] &= \mathcal{N}\int \mathfrak{D}\phi\;\exp\left\{\mathrm{i}\,S\,[\,\phi\,] + \mathrm{i}\int \mathrm{d}x\;\phi(x)\,J(x)\right\} \\ &= \mathcal{N}\int \mathfrak{D}\phi\;\exp\left\{-\,\mathrm{i}\,V\,[\phi]\right\}\,\exp\left\{\mathrm{i}\,S_0\,[\,\phi\,] + \mathrm{i}\int \mathrm{d}x\;\phi(x)\,J(x)\right\} \\ &\overset{\mathrm{def}}{=} \exp\left\{-\,\mathrm{i}\,V\,[\,\delta\,/\,\mathrm{i}\,\delta\,J\,]\right\}\;Z_0\,[\,J\,] \end{aligned} \tag{1.4}$$

To go one step further it is convenient to define

$$Z[J] = \exp\{\mathrm{i}W[J]\} \qquad\qquad Z_0[J] = \exp\{\mathrm{i}W_0[J]\}$$

and thereby

$$\begin{aligned} Z[J] &= \exp\{\mathrm{i}W_0[J]\}\ \exp\{-\mathrm{i}W_0[J]\}\ \exp\{-\mathrm{i}V[\delta/\mathrm{i}\delta J]\}\ Z_0[J] \\ &= \exp\{\mathrm{i}W_0[J]\}\ \Big[1\ +\ \exp\{-\mathrm{i}W_0[J]\}\ \times \\ &\quad \Big(\exp\{-\mathrm{i}V[\delta/\mathrm{i}\delta J]\} - 1\Big)\exp\{\mathrm{i}W_0[J]\}\ \Big] \end{aligned}$$

Taking the logarithm of the above relation we find

$$\mathrm{i}W[J] = \mathrm{i}W_0[J] + \ln\Big(1 + X[J]\Big) \tag{1.5}$$

$$X = e^{-\mathrm{i}W_0}\left(e^{-\mathrm{i}V} - 1\right)e^{\mathrm{i}W_0} \tag{1.6}$$

By expanding $\ln(1+X)$ in Taylor's series, on the one side we obtain

$$\mathrm{i}W = \mathrm{i}W_0 + X - \frac{1}{2}X^2 + \frac{1}{3}X^3 - \ \cdots = \mathrm{i}W_0 + \sum_{k=1}^{\infty}(-1)^{k+1}\,\frac{X^k}{k} \tag{1.7}$$

On the other side, we can in turn expand the dimensionless quantity X as a power series of the dimensionless small coupling parameter $0 \le \lambda < 1$ so that we can write

$$X = \lambda X_1 + \lambda^2 X_2 + \lambda^3 X_3 + \ \cdots \tag{1.8}$$

in such a manner that we finally come to the formal expansion

$$\begin{aligned} \mathrm{i}W &= \mathrm{i}W_0 + \Big(\lambda X_1 + \lambda^2 X_2 + \lambda^3 X_3\ \cdots\Big) - \frac{1}{2}\Big(\lambda X_1 + \lambda^2 X_2 + \ \cdots\Big)^2 \\ &+ \frac{1}{3}\Big(\lambda X_1 + \lambda^2 X_2 + \ldots\Big)^3 + \ \cdots \\ &= \mathrm{i}W_0 + \lambda X_1 + \lambda^2\left(X_2 - \frac{1}{2}X_1^2\right) + \lambda^3\left(X_3 - X_2X_1 + \frac{1}{3}X_1^3\right) \\ &- \ \cdots\cdots\ = \ \mathrm{i}W_0 + \sum_{n=1}^{\infty}\lambda^n Y_n \end{aligned} \tag{1.9}$$

$$\begin{aligned} Y_1 &= X_1 \\ Y_2 &= X_2 - \frac{1}{2}X_1^2 \\ Y_3 &= X_3 - X_2X_1 + \frac{1}{3}X_1^3 \\ &\ \vdots \end{aligned}$$

the dimensionless coefficients Y_n being the so called *connected* parts of the related quantities X_n $(n \in \mathbb{N})$. Hence, within the perturbative approach, the quantity $Z[J]$ will provide the generating functional for the full Green's function of the interacting theory, while the functional $W[J] = -\mathrm{i} \ln Z[J]$ will generate the *connected Green's functions*. The attentive reader should certainly gather the analogy with statistical thermodynamics. As a matter of fact, the partition function Z and the Helmoltz' free energy F do fulfill a very close relation in units of kT, k being the Boltzmann' constant and T the (absolute) temperature : namely, $\beta F = -\ln Z$ $(\beta = 1/kT = 1)$. This means that, after transition to the euclidean formulation, we can identify the generating functional $Z_E[J_E]$ with the canonical partition function and the functional $W_E[J_E] = -\ln Z_E[J_E]$ with the Helmoltz' free energy at some given temperature of equilibrium, in natural units $\beta = 1$. Hence the euclidean Green's functions will correspond to the correlation function of the corresponding mechanical system in thermodynamic equilibrium with a heat reservoir at unit temperature. The above analogy does provide the bridge to formulate and develop the statistical theory of the phase transitions in terms of the very same concptual and mathematical tools which lie on the ground of relativistic quantum field theory.

Turning back to the definitions (1.6), (1.7) and (1.8) it is convenient to introduce the following shorter notations : namely,

$$\begin{aligned} X_1[J] &= -\frac{\mathrm{i}}{4!}\, e^{-\mathrm{i}W_0[J]} \int \mathrm{d}x\, \frac{\delta^4}{\delta J_x^4}\, e^{\mathrm{i}W_0[J]} = Y_1[J] \\ X_2[J] &= \frac{1}{2}\left(-\frac{\mathrm{i}}{4!}\right) e^{-\mathrm{i}W_0[J]} \int \mathrm{d}y\, \frac{\delta^4}{\delta J_y^4}\, Z_0[J]\, X_1[J] \qquad (1.10) \\ &\;\;\vdots \end{aligned}$$

$$W_0[J] = \frac{\mathrm{i}}{2} \int \mathrm{d}x \int \mathrm{d}y\; J(x)\, D_F(x-y)\, J(y) \equiv \frac{\mathrm{i}}{2} \langle\, J_x\, D_{xy}\, J_y \,\rangle$$

Hence we can write

$$X_1 = Y_1 = \left(-\frac{\mathrm{i}}{4!}\right) e^{-\mathrm{i}W_0[J]} \int \mathrm{d}z\, \frac{\delta^4}{\delta J_z^4}\, \exp\left\{ -\frac{1}{2} \langle\, J_x\, D_{xy}\, J_y \,\rangle \right\}$$

Let me evaluate this expression: first we find

$$e^{-\mathrm{i}W_0[J]}\, \frac{\delta}{\delta J_z}\, Z_0[J] = -\langle\, D_{zx}\, J_x \,\rangle$$

$$e^{-\mathrm{i}W_0[J]}\, \frac{\delta^2}{\delta J_z^2}\, Z_0[J] = -D_F(0) + \langle\, D_{zx}\, D_{zy}\, J_x\, J_y \,\rangle$$

$$e^{-\mathrm{i}W_0[J]}\, \frac{\delta^3}{\delta J_z^3}\, Z_0[J] = 3\, D_F(0)\, \langle\, D_{zx}\, J_x \,\rangle - \langle\, D_{zx}\, D_{zy}\, D_{zw}\, J_x\, J_y\, J_w \,\rangle$$

so that we finally obtain

$$
\begin{aligned}
\mathrm{i}Y_1 &= \frac{1}{4!}\, e^{-\mathrm{i}W_0} \int \mathrm{d}x\, \frac{\delta^4}{\delta J_x^4}\, Z_0 = \frac{1}{8} \int \mathrm{d}x\ D_F^2(0) \\
&- \frac{1}{4} \int \mathrm{d}x\ D_F(0) \int \mathrm{d}x_1 \int \mathrm{d}x_2\ D_F(x-x_1)\, D_F(x-x_2)\, J(x_1)\, J(x_2) \\
&+ \frac{1}{24} \int \mathrm{d}x\ \prod_{k=1}^{4} \int \mathrm{d}x_k\ D_F(x-x_k)\, J(x_k) \\
Y_1 &= \left(-\frac{\mathrm{i}}{4!}\right) \int \mathrm{d}x\ \Big(3\, D_F^2(0) - 6\, D_F(0)\, \langle\, D_{x1}\, D_{x2}\, J_1\, J_2 \,\rangle \\
&+ \langle\, D_{x1}\, D_{x2}\, D_{x3}\, D_{x4}\, J_1\, J_2\, J_3\, J_4 \,\rangle \Big) \qquad (1.11)
\end{aligned}
$$

which leads to the first order $O(\lambda)$ correction to the generating functional of the connected Green's functions

$$
\begin{aligned}
\mathrm{i}W &\approx \mathrm{i}W_0 + \lambda\, Y_1 \\
&= \mathrm{i}W_0 - \frac{\mathrm{i}\lambda}{4!} \int \mathrm{d}x\ \Big(3\, D_F^2(0) - 6\, D_F(0)\, \langle\, D_{x1}\, D_{x2}\, J_1\, J_2 \,\rangle \\
&+ \langle\, D_{x1}\, D_{x2}\, D_{x3}\, D_{x4}\, J_1\, J_2\, J_3\, J_4 \,\rangle \Big) \qquad (1.12)
\end{aligned}
$$

Next we have to calculate X_2. The obvious generalization of the symbolic Leibnitz' chain rule to the functional differentiation reads

$$
\begin{aligned}
\frac{\delta^n}{\delta J^n}\,(\, Z_0\, X_1\,) &= (\, Z_0 + X_1\,)^{(n)} \\
&= Z_0\, \frac{\delta^n}{\delta J^n}\, X_1 + \binom{n}{1} \left(\frac{\delta}{\delta J}\, Z_0\right) \frac{\delta^{n-1}}{\delta J^{n-1}}\, X_1 \\
&+ \binom{n}{2} \left(\frac{\delta^2}{\delta J^2}\, Z_0\right) \frac{\delta^{n-2}}{\delta J^{n-2}}\, X_1 + \cdots + \left(\frac{\delta^n}{\delta J^n}\, Z_0\right) X_1
\end{aligned}
$$

and in particular

$$
\begin{aligned}
\frac{\delta^4}{\delta J^4}\,(\, Z_0\, X_1\,) &= Z_0\, \frac{\delta^4}{\delta J^4}\, X_1 + 4 \left(\frac{\delta}{\delta J}\, Z_0\right) \frac{\delta^3}{\delta J^3}\, X_1 \\
+\, 6 \left(\frac{\delta^2}{\delta J^2}\, Z_0\right) \frac{\delta^2}{\delta J^2}\, X_1 &+ 4 \left(\frac{\delta^3}{\delta J^3}\, Z_0\right) \frac{\delta}{\delta J}\, X_1 + \left(\frac{\delta^4}{\delta J^4}\, Z_0\right) X_1
\end{aligned}
$$

Then, from equation (1.10) we readily get

$$X_2 = \frac{1}{2} Y_1^2 + Y_2$$

$$\begin{aligned} Y_2 &= \frac{1}{2}\left(-\frac{\mathrm{i}}{4!}\right) e^{-\mathrm{i}W_0} \int \mathrm{d}z \, \left\{ Z_0 \frac{\delta^4}{\delta J_z^4} + 4\left(\frac{\delta}{\delta J_z} Z_0\right) \frac{\delta^3}{\delta J_z^3} \right. \\ &+ \left. 6\left(\frac{\delta^2}{\delta J_z^2} Z_0\right) \frac{\delta^2}{\delta J_z^2} + 4\left(\frac{\delta^3}{\delta J_z^3} Z_0\right) \frac{\delta}{\delta J_z} \right\} Y_1[J] \qquad (1.13) \end{aligned}$$

Taking the functional derivatives of equation (1.11) we find

$$\begin{aligned} \frac{\delta}{\delta J_z} Y_1[J] &= \left(-\frac{\mathrm{i}}{4!}\right) \int \mathrm{d}x \, \Big(-12 D_F(0) D_F(x-z) \langle D_{x1} J_1 \rangle \\ &+ 4 D_F(x-z) \langle D_{x1} D_{x2} D_{x3} J_1 J_2 J_3 \rangle \Big) \\ \frac{\delta^2}{\delta J_z^2} Y_1[J] &= \frac{1}{2\mathrm{i}} \int \mathrm{d}x \; D_F^2(x-z) \Big(-D_F(0) + \langle D_{x1} D_{x2} J_1 J_2 \rangle \Big) \\ \frac{\delta^3}{\delta J_z^3} Y_1[J] &= \frac{1}{\mathrm{i}} \int \mathrm{d}x \; D_F^3(x-z) \langle D_{x1} J_1 \rangle \\ \frac{\delta^4}{\delta J_z^4} Y_1[J] &= \frac{1}{\mathrm{i}} \int \mathrm{d}x \; D_F^4(x-z) \qquad (1.14) \end{aligned}$$

so that explicit term-by-term evaluation yields

$$\frac{1}{2}\left(-\frac{\mathrm{i}}{4!}\right) \int \mathrm{d}z \; \frac{\delta^4}{\delta J_z^4} Y_1[J] = \frac{-1}{2(4!)} \int \mathrm{d}x \int \mathrm{d}y \; D_F^4(x-y)$$

$$\begin{aligned} &\left(-\frac{\mathrm{i}}{4!}\right) \frac{2}{Z_0} \int \mathrm{d}z \; \left(\frac{\delta}{\delta J_z} Z_0\right) \frac{\delta^3}{\delta J_z^3} Y_1 = \\ &\frac{4}{2(4!)} \int \mathrm{d}x \int \mathrm{d}y \; D_F^3(x-y) \langle D_{x1} D_{y2} J_1 J_2 \rangle \end{aligned}$$

$$\begin{aligned} &\frac{1}{2}\left(-\frac{\mathrm{i}}{4!}\right) \frac{6}{Z_0} \int \mathrm{d}z \; \left(\frac{\delta^2}{\delta J_z^2} Z_0\right) \frac{\delta^2}{\delta J_z^2} Y_1 = \\ &\frac{3}{2}\left(-\frac{1}{4!}\right) \int \mathrm{d}x \int \mathrm{d}y \; D_F^2(x-y) \times \\ &\Big(D_F^2(0) - 2 D_F(0) \langle D_{x1} D_{x2} J_1 J_2 \rangle + \langle D_{x1} D_{x2} D_{x3} D_{x4} J_1 J_2 J_3 J_4 \rangle \Big) \end{aligned}$$

$$
\frac{1}{2}\left(-\frac{\mathrm{i}}{4!}\right)\frac{4}{Z_0}\int \mathrm{d}z\ \left(\frac{\delta^3}{\delta J_z^3}\,Z_0\right)\ \frac{\delta Y_1}{\delta J_z}
$$

$$
= \frac{1}{2}\left(-\frac{\mathrm{i}}{4!}\right)^2\cdot 4\cdot 4\int \mathrm{d}x\int \mathrm{d}y\ D_F(x-y)\ \times
$$

$$
\Big(-\,3\,D_F(0)\,\langle\, D_{x1}\,J_1\,\rangle + \langle\, D_{x1}\,D_{x2}\,D_{x3}\,J_1\,J_2\,J_3\,\rangle\Big)
$$

$$
\times\ \Big(3\,D_F(0)\,\langle\, D_{y2}\,J_2\,\rangle - \langle\, D_{y4}\,D_{y5}\,D_{y6}\,J_4\,J_5\,J_6\,\rangle\Big)
$$

It follows that the lowest order result for the source-independent quantity

$$
\begin{aligned}
\mathrm{i}W[\,0\,] \ &\approx\ \lambda\,Y_1[\,0\,]+\lambda^2\,Y_2[\,0\,]\ =\ -\frac{\mathrm{i}\lambda}{8}\int \mathrm{d}x\ D_F^2(0)\\
&-\ \frac{\lambda^2}{48}\int \mathrm{d}x\int \mathrm{d}y\ D_F^2(x-y)\ \Big(3\,D_F^2(0)+D_F^2(x-y)\Big)
\end{aligned}
$$

appears to be a divergent correction to the cosmological constant due to the real scalar field self-interaction. Moreover, the remaining source-dependent lowest order contribution take the form

$$
\begin{aligned}
Y_2[\,J\,]\ &=\ \frac{1}{12}\int \mathrm{d}x\int \mathrm{d}y\ D_F^3(x-y)\,\langle\, D_{x1}\,D_{y2}\,J_1\,J_2\,\rangle\\
&+\ \frac{1}{8}\int \mathrm{d}x\int \mathrm{d}y\ D_F^2(x-y)D_F(0)\,\langle\, D_{x1}\,D_{y2}\,J_1\,J_2\,\rangle\\
&+\ \frac{1}{8}\int \mathrm{d}x\int \mathrm{d}y\ D_F(x-y)\,D_F^2(0)\,\langle\, D_{x1}\,D_{y2}\,J_1\,J_2\,\rangle\\
&-\ \frac{3}{2\,(4!)}\int \mathrm{d}x\int \mathrm{d}y\ D_F^2(x-y)\ \langle\, D_{x1}\,D_{x2}\,D_{y3}\,D_{y4}\,J_1\,J_2\,J_3\,J_4\,\rangle\\
&-\ \frac{2}{4!}\int \mathrm{d}x\int \mathrm{d}y\ D_F(x-y)\,D_F(0)\ \langle\, D_{x1}\,D_{y2}\,D_{y3}\,D_{y4}\,J_1\,J_2\,J_3\,J_4\,\rangle\\
&+\ \frac{1}{2\,(3!)^2}\int \mathrm{d}x\int \mathrm{d}y\ D_F(x-y)\\
&\times\ \langle\, D_{x1}\,D_{x2}\,D_{x3}\,D_{y4}\,D_{y5}\,D_{y6}\,J_1\,J_2\,J_3\,J_4\,J_5\,J_6\,\rangle
\end{aligned}
$$

The resulting connected Green's functions follow from the definitions

$$
\begin{aligned}
\mathrm{i}W[\,J\,]\ &\stackrel{\mathrm{def}}{=}\ \sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{n!}\int \mathrm{d}x_1\,J(x_1)\cdots\int \mathrm{d}x_n\,J(x_n)\\
&\times\ G_c^{(n)}(x_1,\cdots,x_n)
\end{aligned}
\qquad (1.15)
$$

$$
G_c^{(n)}(x_1,\cdots,x_n)\ \stackrel{\mathrm{def}}{=}\ (-\,\mathrm{i}\,)^{\,n-1}\,\delta^{(n)}\,W[\,J\,]\,/\,\delta\,J(x_1)\cdots\delta\,J(x_n)\,\Big|_{J=0}
$$

The perturbative expansion of the $n-$point connected Green's function does follow directly from the series expansion (1.9). Here below we list the lowest order 2-point connected Green's function, which is usually named the full propagator, as well as the 4-point and 6-point connected Green's functions : namely,

$$
\begin{aligned}
G_c^{(2)}(x_1 - x_2) &= G_0^{(2)}(x_1 - x_2) - \sum_{n=0}^{\infty} \left. \frac{\lambda^n \, \delta^{(2)} Y_n}{\delta J(x_1) \, \delta J(x_2)} \right\rfloor_{J=0} \\
&= D_F(x_1 - x_2) \\
&- \frac{\mathrm{i}\lambda}{2} \int \mathrm{d}y \; D_F(x_1 - y) D_F(0) D_F(y - x_2) \\
&- \frac{\lambda^2}{6} \int \mathrm{d}x \int \mathrm{d}y \; D_F(x_1 - x) D_F^3(x - y) D_F(y - x_2) \\
&- \frac{\lambda^2}{4} \int \mathrm{d}x \int \mathrm{d}y \; D_F(x_1 - x) D_F(0) D_F^2(x - y) D_F(y - x_2) \\
&- \frac{\lambda^2}{4} \int \mathrm{d}x \int \mathrm{d}y \; D_F(x_1 - x) D_F^2(0) D_F(x - y) D_F(y - x_2) \\
&+ O(\lambda^3)
\end{aligned}
\tag{1.16}
$$

$$
\begin{aligned}
&G_c^{(4)}(x_1, x_2, x_3, x_4) = \sum_{n=0}^{\infty} \left. \frac{\lambda^n \, \delta^{(4)} Y_n}{\delta J(x_1) \, \delta J(x_2) \, \delta J(x_3) \, \delta J(x_4)} \right\rfloor_{J=0} \\
&- \mathrm{i}\lambda \int \mathrm{d}x \; D_F(x_1 - x) D_F(x_2 - x) D_F(x_3 - x) D_F(x_4 - x) \\
&- \frac{\lambda^2}{2} \int \mathrm{d}x \int \mathrm{d}y \; D_F^2(x - y) \; \times \\
&\Big[D_F(x_1 - x) D_F(x_2 - x) D_F(x_3 - y) D_F(x_4 - y) \\
&\quad + D_F(x_1 - x) D_F(x_3 - x) D_F(x_2 - y) D_F(x_4 - y) \\
&\quad + D_F(x_1 - x) D_F(x_4 - x) D_F(x_2 - y) D_F(x_3 - y) \Big] \\
&- \frac{\lambda^2}{2} \int \mathrm{d}x \int \mathrm{d}y \; D_F(x - y) D_F(0) \; \times \\
&\Big[D_F(x_1 - x) D_F(x_2 - x) D_F(x_3 - x) D_F(x_4 - y) \\
&+ \text{cyclic perm.} \Big] \; + \; O(\lambda^3)
\end{aligned}
\tag{1.17}
$$

and finally

$$
G_c^{(6)}(x_1, \ldots, x_6) = -\lambda^2 \int \mathrm{d}x \int \mathrm{d}y \; D_F(x - y) \; \times
$$

$$\sum_{(\imath\jmath\kappa)} D_F(x_\imath - x)\, D_F(x_\jmath - x)\, D_F(x_\kappa - x)$$
$$\times\, D_F(x_\ell - x)\, D_F(x_m - x)\, D_F(x_n - x) + O(\lambda^3) \qquad (1.18)$$

where the sum in the last expression runs over the triples $(\imath\jmath\kappa)$ in which $\imath < \jmath < \kappa$, with $\imath, \jmath, \kappa = 1, 2, \ldots, 6$, while the triples $(\ell m n)$ take the complementary values, *i.e.* , $(\ell m n) = (456)$ when $(\imath\jmath\kappa) = (123)$ *et cetera.* The remaining Green's functions get no contributions up to this order in λ.

The Fourier transformation of the relativistic wave field field functions in the Minkowski four dimensional space-time are defined to be

$$u_A(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, \widetilde{u}_A(k)\, \exp\{-\mathrm{i}k\cdot x\} \qquad (1.19)$$

$$\widetilde{u}_A(k) = \int \mathrm{d}^4 x\; u_A(x)\, \exp\{\mathrm{i}k\cdot x\} \qquad (1.20)$$

where the index $A = 1, 2, \ldots, N$ labels, as usual, the components of the real or complex wave field functions.

The Fourier transforms of the $n-$point Green's functions, connected and disconnected, *i.e.* the momentum space Green's functions, are defined by

$$\widetilde{G}_c^{(n)}(k_1, \ldots, k_n)\, (2\pi)^4\, \delta(k_1 + k_2 + \cdots + k_n) =$$
$$\int \mathrm{d}x_1 \cdots \int \mathrm{d}x_n\; G_c^{(n)}(x_1, \ldots, x_n)\; \exp\{\mathrm{i}k_1\cdot x_1 + \cdots + \mathrm{i}k_n\cdot x_n\} \quad (1.21)$$

in such a manner that the total momentum conservation encoded by the $\delta-$distribution does vindicate the space-time translation invariance. If we remember the momentum space scalar Feynman propagator

$$D_F(k) = \widetilde{G}_0^{(2)}(k, -k) = \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon}$$

it is straightforward to derive the perturbative expansion of the momentum space Green's functions. We find

$$\begin{aligned}
\widetilde{G}_c^{(2)}(k, -k) \;&=\; D_F(k) + \frac{1}{2}(-\mathrm{i}\lambda)\, D_F^2(k) \int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\, \frac{\mathrm{i}}{\ell^2 - m^2 + \mathrm{i}\varepsilon} \\
&+\; \frac{1}{6}(-\mathrm{i}\lambda)^2\, D_F^2(k) \int \frac{\mathrm{d}^4\ell_1}{(2\pi)^4} \int \frac{\mathrm{d}^4\ell_2}{(2\pi)^4} \int \frac{\mathrm{d}^4\ell_3}{(2\pi)^4} \\
&\quad (2\pi)^4\, \delta(k - \ell_1 - \ell_2 - \ell_3) \\
&\times\; \frac{\mathrm{i}}{\ell_1^2 - m^2 + \mathrm{i}\varepsilon} \cdot \frac{\mathrm{i}}{\ell_2^2 - m^2 + \mathrm{i}\varepsilon} \cdot \frac{\mathrm{i}}{\ell_3^2 - m^2 + \mathrm{i}\varepsilon}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{4}(-\mathrm{i}\lambda)^2\, D_F^2(k) \int \frac{d^4\ell}{(2\pi)^4}\, \frac{\mathrm{i}}{\ell^2 - m^2 + \mathrm{i}\varepsilon} \\
&\times \int \frac{d^4\ell_1}{(2\pi)^4} \int \frac{d^4\ell_2}{(2\pi)^4}\, \frac{(2\pi)^4\,\delta(k-\ell_1-\ell_2)}{(\ell_1^2 - m^2 + \mathrm{i}\varepsilon)(\ell_2^2 - \mathrm{m}^2 + \mathrm{i}\varepsilon)} \\
&+ \frac{1}{4}(-\mathrm{i}\lambda)^2\, D_F^3(k) \left(\int \frac{d^4\ell}{(2\pi)^4}\, \frac{\mathrm{i}}{\ell^2 - m^2 + \mathrm{i}\varepsilon} \right)^2 \\
&+ O(\lambda^3) \qquad (1.22)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{G}_c^{(4)}(k_1,k_2,k_3,k_4) &= \prod_{a=1}^{4} \frac{\mathrm{i}}{k_a^2 - m^2 + \mathrm{i}\varepsilon} \Big\{ (-\mathrm{i}\lambda) \\
&+ \frac{1}{2}(-\mathrm{i}\lambda)^2 \sum_{\jmath=1}^{4} \frac{\mathrm{i}}{k_\jmath^2 - m^2 + \mathrm{i}\varepsilon} \int \frac{d^4\ell}{(2\pi)^4}\, \frac{\mathrm{i}}{\ell^2 - m^2 + \mathrm{i}\varepsilon} \\
&+ \frac{1}{2}(-\mathrm{i}\lambda)^2 \int \frac{d^4\ell_1}{(2\pi)^4}\, \frac{\mathrm{i}}{\ell_1^2 - m^2 + \mathrm{i}\varepsilon} \int \frac{d^4\ell_2}{(2\pi)^4}\, \frac{\mathrm{i}}{\ell_2^2 - m^2 + \mathrm{i}\varepsilon} \\
&\times \sum_{(\imath\jmath)} (2\pi)^4\,\delta(\ell_1 + \ell_2 - k_\imath - k_\jmath) + O(\lambda^3) \Big\} \qquad (1.23)
\end{aligned}
$$

where the sum $(\imath\jmath)$ runs over the three pairs $(12), (13), (14)$. Finally

$$
\begin{aligned}
\widetilde{G}_c^{(6)}(k_1,\dots,k_6) &= \Big[\prod_{a=1}^{6} \frac{\mathrm{i}}{k_a^2 - m^2 + \mathrm{i}\varepsilon} \Big] (-\mathrm{i}\lambda)^2 \\
&\times \sum_{(\imath\jmath\kappa)} \frac{\mathrm{i}}{(k_\imath + k_\jmath + k_\kappa)^2 - m^2 + \mathrm{i}\varepsilon} + O(\lambda^3) \qquad (1.24)
\end{aligned}
$$

where again the sum is over the same triples as in equation (1.18).

1.2 Self-interacting real scalar field

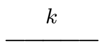
The above expressions are admittedly rather cumbersome and unwieldy. One urgently needs to device some clever code to generate them to any order in perturbation theory. This is precisely what the genious of Richard Patrick Feynman achieved for us : the rules of correspondence that are universally known as *the Feynman rules*

Richard Patrick Feynman
The Theory of Positrons
Phys. Rev. 76, 749 - 759 (1949) [Issue 6 – September 1949]

Space-Time Approach to Quantum Electrodynamics
Phys. Rev. 76, 769 - 789 (1949) [Issue 6 – September 1949]

Keeping in mind the applications to the scattering processes, it is more convenient to express the Feynman rules in momentum space. Hence, we shall represent the Feynman scalar propagator in momentum space by its 4-momentum k and indicating the direction with an arrow. Moreover when four line meets at a vertex, we always understand the momentum flow as incoming towards the vertex. Then we are left with the following *momentum-space Feynman rules* to build up the connected Green's functions to all orders in perturbation theory :

1. For each propagator, see fig. N 1



$$= \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon}$$

2. For each vertex, see fig. N 2

$$\bullet \quad = \quad -\frac{\mathrm{i}\lambda}{4!}$$

3. Impose momentum conservation with all momenta incoming

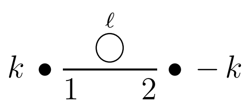
$$k_1 + k_2 + k_3 + k_4 \;=\; 0$$

4. Integrate over each internal momentum, *i.e.* each momentum ℓ which is not an argument of the Green's function

$$\int \frac{\mathrm{d}^4 \ell}{(2\pi)^4}$$

5. In order to get the contribution to $\widetilde{G}_c^{(n)}(k_1, \ldots, k_n)$, draw all possible arrangements which are topologically inequivalent, after identification of the so called external legs corresponding to the external momenta $k_1, \ldots, k_n$. The number of ways a given diagram can be drawn is the topological weight of the diagram. The symmetry factor of the diagram is equal to its topological weight divided by 4!

   To give an example, consider the diagram of fig. N 3



We need one vertex and three propagators. There are four ways to attach the first propagator from 1 to the vertex, three ways to attach the second propagator from 2 to the vertex. Hence the topological weight is $4 \cdot 3$ and the symmetry factor $4 \cdot 3/4! = \frac{1}{2}$. Other 2−loop examples are illustrated in fig. N 4 and in fig. N 5. Let ℓ the internal momentum circulating around the closed loop: then the rules give

$$\frac{1}{2}(-\mathrm{i}\lambda)\,\frac{\mathrm{i}}{k^2-m^2+\mathrm{i}\varepsilon}\quad\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\,\frac{\mathrm{i}}{\ell^2-m^2+\mathrm{i}\varepsilon}\cdot\frac{\mathrm{i}}{k^2-m^2+\mathrm{i}\varepsilon}$$

Thus we see that the Feynman rules reduced essentially the problem of setting up the perturbative expansion of the Green's functions to that faced by a child assembling a *Lego* set. More important, the structure of the propagator and of the vertex, *i.e.* the main tools of the game, can be directly read off of the classical Lagrange density. Consider in fact the functional integral representation of the generating functional

$$\begin{aligned} Z[\,J\,] &= \exp\{\,\mathrm{i}W[\,J\,]\} \\ &= \mathcal{N}\int\mathfrak{D}\phi\ \exp\left\{\mathrm{i}S_0[\,\phi\,]-\,\mathrm{i}V[\,\phi\,]+\mathrm{i}\int\mathrm{d}x\ \phi(x)\,J(x)\right\} \end{aligned} \tag{1.25}$$

and let me focuss on the first two addenda in the exponent, *viz.*,

$$\mathrm{i}S_0[\,\phi\,]=\mathrm{i}\int\mathrm{d}x\ \tfrac{1}{2}\,\{g^{\mu\nu}\,\partial_\mu\phi\,(x)\,\partial_\nu\phi\,(x)-m^2\,\phi^2\,(x)\}$$

$$-\,\mathrm{i}\,V[\,\phi\,]=-\,\frac{\mathrm{i}\lambda}{4!}\int\mathrm{d}x\ \phi^4(x)$$

Taking the Fourier transform after a partial integration we come to

$$\mathrm{i}S_0[\,\phi\,]=\int\frac{dk}{(2\pi\,)^4}\,\tfrac{1}{2}\,\widetilde{\phi}(-k)\,\mathrm{i}\,(\,k^2-m^2\,)\,\widetilde{\phi}\,(k)$$

$$-\,\mathrm{i}\,V[\,\phi\,]=-\,\frac{\mathrm{i}\lambda}{4!}\,\prod_{\jmath=1}^{4}\int\frac{dk_\jmath}{(2\pi\,)^4}\,\widetilde{\phi}\,(k_\jmath)\,(\,2\pi\,)^4\,\delta\,(\,k_1+k_2+k_3+k_4\,)$$

whence it appears quite manifest that the momentum space Feynman rules 1.− 3. can be directly read off of the classical action. More precisely,

- the Feynman propagator is just equal to the opposite of the inverse of the kinetic operator, *viz.* the Klein-Gordon operator $(\,k^2-m^2\,)$ in the present scalar field case
- the vertex and the overall four momentum conservation are evidently encoded within the classical interaction potential in momentum space

1.3 Yukawa theory

The Feynman rules for spinor fields can be rather easily gathered from the paradigmatic simple model known as the Yukawa theory

Hideki Yukawa
On the Interaction of Elementary Particles. I
Supplement of the Progress of theoretical physics No. 1 (1935) pp. 1-10

The Yukawa model involves a real scalar field interacting with a complex Dirac spinor field, the classical Lagrange density being given by

$$\mathcal{L}_{\text{Yukawa}} = \tfrac{1}{2}\, g^{\mu\nu}\, \partial_\mu \phi\, \partial_\nu \phi - \tfrac{1}{2}\, m^2\, \phi^2 + \tfrac{1}{2}\, \bar{\psi}\, \gamma^\mu\, \mathrm{i}\, \overleftrightarrow{\partial}_\mu \psi - (\, M + g\,\phi\,)\; \bar{\psi}\psi$$

where g is the dimensionless Yukawa coupling parameter. Turning to the momentum space we find

$$\begin{aligned}
\mathrm{i}\, S_{\text{Yukawa}}\,[\,\phi, \psi, \bar{\psi}\,] \;&=\; \mathrm{i} \int \mathrm{d}x\; \mathcal{L}_{\text{Yukawa}}\,[\,\phi, \psi, \bar{\psi}\,] \\
=\;& \frac{\mathrm{i}}{2} \int \frac{\mathrm{d}k}{(\,2\pi\,)^4}\; \widetilde{\phi}(-k)\, \left(k^2 - m^2\right)\, \widetilde{\phi}(k) \\
+\;& \mathrm{i} \int \frac{\mathrm{d}p}{(\,2\pi\,)^4}\; \widetilde{\psi}^{\,\dagger}(p)\, \gamma^0\, (\,\not{p} - M\,)\, \widetilde{\psi}(p) - \mathrm{i}\, g \int \frac{\mathrm{d}k}{(\,2\pi\,)^4} \int \frac{\mathrm{d}p}{(\,2\pi\,)^4} \int \frac{\mathrm{d}q}{(\,2\pi\,)^4} \\
& \widetilde{\phi}(k)\, \widetilde{\psi}^{\,\dagger}(q)\, \gamma^0\, \widetilde{\psi}(p)\; (2\pi\,)^4\, \delta\,(\,k + p - q\,)
\end{aligned}$$

in such a manner that the scalar and spinor Feynman propagators read

$$\overset{k}{\text{———}} \quad = \quad \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon}$$

$$\text{– – –}\overset{p}{\blacktriangleright}\text{– – –} \quad = \quad \frac{\mathrm{i}\,(\,\not{p} + M\,)}{p^2 - M^2 + \mathrm{i}\varepsilon} = \frac{\mathrm{i}}{\not{p} - M}$$

which correspond, as explained before, to the opposite of the inverse of the Klein-Gordon and Dirac kinetic operators respectively. On the other side, the vertex is clearly given by $-\,\mathrm{i}\,g$, while the energy-momentum conservation here becomes $k + p - q \;=\; 0$. Several comments are now in order.

1. It is worthwhile to notice that the direction of the energy-momentum on a fermion line is always significant. On a fermion propagator, the 4-momentum must be assigned in the direction of the charge flow. Here, the momentum of a particle is always taken towards the vertex, the particles being assumed to carry *negative elementary charge* $-\,e$ with $e \,>\, 0$. Hence, the direction of the momentum is always understood

by definition to follow the negative charge flow. Thus, the momentum of the antiparticle will go out of the vertex, which corresponds to the term $-q$ in the argument of the energy-momentum $\delta-$distribution.

2. The diagrams of the Yukawa theory never exhibit topological weights nor symmetry factors, since the three fields $(\phi\bar{\psi}\psi)$ in the interaction Lagrangian can not be interchanged one another.

3. Finally, the Grassmann nature of the spinor field is reflected in one crucial change in the Feynman rules : whenever a *closed fermion line*, which is usually named a *fermion cycle* or more commonly a *fermion loop*, appears in a diagram, then one must multiply the diagram by a factor (-1) for each fermion loop of the diagram. This latter rule can be illustrated by the following enlightening example.

1.3.1 Fermion determinants

Consider the generating functional for the Yukawa field theory

$$Z[\zeta, \bar{\zeta}, J] \stackrel{\text{def}}{=} \mathcal{N}\int\mathfrak{D}\phi\int\mathfrak{D}\psi\int\mathfrak{D}\bar{\psi}\ \exp\left\{\mathrm{i}\,S_{\text{Yukawa}}[\bar{\psi}, \psi, \phi]\right\}$$
$$\times\ \exp\left\{\mathrm{i}\int\mathrm{d}x\ \left[\bar{\zeta}(x)\,\psi(x)+\bar{\psi}(x)\,\zeta(x)+J(x)\,\phi(x)\right]\right\} \qquad (1.26)$$

By making use of the same trick I have employed before in the case of the perturbative definition of the generating functional for the self-interacting real scalar field theory I can write

$$Z[J, \zeta, \bar{\zeta}] = \exp\{-\mathrm{i}\,V[\delta/\mathrm{i}\,\delta J, \delta/\mathrm{i}\,\delta\bar{\zeta}, \delta/\mathrm{i}\,\delta\zeta]\}\ Z_0[\zeta, \bar{\zeta}, J]$$
$$V = g\int\mathrm{d}x\ (\delta/\mathrm{i}\,\delta J(x))\,(\delta/\mathrm{i}\,\delta\zeta_\alpha(x))\,\big(\delta/\mathrm{i}\,\delta\bar{\zeta}_\alpha(x)\big) \qquad (1.27)$$

while

$$Z_0[\zeta, \bar{\zeta}, J] = Z_0^{\mathrm{F}}[\zeta, \bar{\zeta}]\cdot Z_0^{\mathrm{B}}[J] = \exp\{\mathrm{i}W_0^{\mathrm{B}}[J]+\mathrm{i}W_0^{\mathrm{F}}[\zeta, \bar{\zeta}]\}$$
$$Z_0^{\mathrm{F}}[\zeta, \bar{\zeta}] = \exp\left\{-\int\mathrm{d}x\int\mathrm{d}y\ \bar{\zeta}(x)\,S_F(x-y)\,\zeta(y)\right\}$$
$$Z_0^{\mathrm{B}}[J] = \exp\left\{-\tfrac{1}{2}\int\mathrm{d}x\int\mathrm{d}y\ J(x)\,D_F(x-y)\,J(y)\big]\right\}$$

It is important to remark that in the definition (1.27), where I have explicitly written the repeated summed spinor indices for the sake of clarity, the order of the anticommuting Grassmann-like functional derivatives is crucial.

Now, to our task, it turns out to be convenient to rewrite the generating functional in the following equivalent form : namely,

$$
\begin{aligned}
Z[\,J\,,\,\zeta\,,\,\bar\zeta\,] \;=\; \mathcal{N}' \int \mathfrak{D}\phi\ \exp\left\{\mathrm{i}\,S_0[\,\phi\,]+\mathrm{i}\int \mathrm{d}y\ J\,(y)\,\phi\,(y)\right\} \\
\times\ \exp\left\{-\,\mathrm{i}\,g\int \mathrm{d}x\ \phi\,(x)\left(\delta^{(2)}/\,\mathrm{i}\,\delta\,\zeta_{\,x}\,\mathrm{i}\,\delta\,\bar\zeta_{\,x}\right)\right\}\ Z_0^{\mathrm{F}}[\,\zeta\,,\,\bar\zeta\,]
\end{aligned}
$$

where $S_0\,[\,\phi\,]$ denotes the classical action for the free real scalar field. As a consequence, if we set

$$
\begin{aligned}
\exp\{-\,\mathrm{i}\,V_\phi[\,\delta/\,\mathrm{i}\,\delta\,\bar\zeta\,,\,\delta/\,\mathrm{i}\,\delta\,\zeta\,]\,\} \;\equiv \\
\exp\left\{-\,\mathrm{i}\,g\int \mathrm{d}x\ \phi\,(x)\left(\delta^{(2)}/\,\mathrm{i}\,\delta\,\zeta_{\,x}\,\mathrm{i}\,\delta\,\bar\zeta_{\,x}\right)\right\} \\
Z_\phi^{\mathrm{F}}[\,\zeta\,,\,\bar\zeta\,] \;=\; \exp\{-\,\mathrm{i}\,V_\phi[\,\delta/\,\mathrm{i}\,\delta\,\bar\zeta\,,\,\delta/\,\mathrm{i}\,\delta\,\zeta\,]\,\}\ Z_0^{\mathrm{F}}[\,\zeta\,,\,\bar\zeta\,] \qquad (1.28)
\end{aligned}
$$

we obtain the functional integral representation

$$
\begin{aligned}
Z_\phi^{\mathrm{F}}[\,\zeta\,,\,\bar\zeta\,] \;=\; \mathcal{N}_\phi \int \mathfrak{D}\psi \int \mathfrak{D}\bar\psi \\
\exp\left\{\,\mathrm{i}\int \mathrm{d}x\ \bar\psi\,(x)\left[\,\tfrac{1}{2}\,\gamma^\mu\,\mathrm{i}\,\overleftrightarrow{\partial}_{\!\mu}-M-g\,\phi\,(x)\right]\psi\,(x)\,\right\} \\
\times\ \exp\left\{\mathrm{i}\int \mathrm{d}x\ \left[\,\bar\zeta(x)\,\psi\,(x)+\bar\psi\,(x)\,\zeta(x)\,\right]\right\} \qquad (1.29)
\end{aligned}
$$

with

$$
\begin{aligned}
Z_\phi^{\mathrm{F}}[\,0\,,\,0\,] \;=\; \mathcal{N}_\phi \int \mathfrak{D}\psi \int \mathfrak{D}\bar\psi \\
\exp\left\{\,\mathrm{i}\int \mathrm{d}x\ \bar\psi\,(x)\left[\,\tfrac{1}{2}\,\gamma^\mu\,\mathrm{i}\,\overleftrightarrow{\partial}_{\!\mu}-M-g\,\phi\,(x)\right]\psi\,(x)\,\right\} \\
\stackrel{\mathrm{def}}{=}\ \mathcal{N}_\phi\,\det\,\|\ \mathrm{i}\,\partial\!\!\!/-M-g\,\phi\ \| \qquad (1.30)
\end{aligned}
$$

the latter definition being understood, as usual, up to the Wick rotation to the euclidean space. Notice that the constant $\mathcal{N}_\phi$ is conveniently fixed by the requirement that in the limit $\phi\to 0$ we recover $Z_0^{\mathrm{F}}[\,0\,,\,0\,]=1\,.$

As a consequence, we eventually come to the symbolic equalities

$$
\begin{aligned}
\frac{\det\,\|\ \mathrm{i}\,\partial\!\!\!/-M-g\,\phi\ \|}{\det\,\|\ \mathrm{i}\,\partial\!\!\!/-M\ \|} &= \det\,\|\ \mathbb{I}-g\,(\,\mathrm{i}\,\partial\!\!\!/-M\,)^{-1}\,\phi\ \| \qquad (1.31) \\
&= \exp\{\,\mathrm{Tr}\ \ln\,\|\ \mathbb{I}-g\,(\,\mathrm{i}\,\partial\!\!\!/-M\,)^{-1}\,\phi\ \|\,\} \\
&= \exp\left\{(-1)\sum_{n=1}^{\infty}\,\frac{1}{n}\ \mathrm{Tr}\,[\,(\,\mathrm{i}\,\partial\!\!\!/-M\,)^{-1}\,g\,\phi\,]^{\,n}\,\right\}
\end{aligned}
$$

in which we understand

$$\langle x | (\mathrm{i}\,\partial\!\!\!/ - M)^{-1} | y \rangle \;=\; \frac{1}{\mathrm{i}}\, S_F(x-y)$$

while the symbol [Tr] indicates the sum over spinor indices and integration over space-time coordinates. After setting

$$X_\phi^n = (-1)\,\mathrm{Tr}\,[\,(\mathrm{i}\,\partial\!\!\!/ - M)^{-1}\, g\,\phi\,]^n$$

explicit evaluation for $n = 1$ yields

$$\begin{aligned} X_\phi^1 &= (-1)\,\mathrm{Tr}\,[\,(\mathrm{i}\,\partial\!\!\!/ - M)^{-1}\, g\,\phi\,] \\ &= -g \int \mathrm{d}x_1\; \langle x_1 | (\mathrm{i}\,\partial\!\!\!/ - M)^{-1}\, \phi\, | x_1 \rangle \\ &= -g \int \mathrm{d}x_1 \int \mathrm{d}y\; \langle x_1 | (\mathrm{i}\,\partial\!\!\!/ - M)^{-1} | y \rangle \langle y | \phi | x_1 \rangle \\ &= \mathrm{i}g \int \mathrm{d}x_1 \int \mathrm{d}y\; \phi(x_1)\, \delta(x_1 - y)\; \mathrm{tr}\, S_F(x_1 - y) \\ &= \mathrm{i}g \int \mathrm{d}x_1\; \phi(x_1)\; \mathrm{tr}\, S_F(x_1 - x_1) \\ &= \mathrm{i}g \int \mathrm{d}x_1\; \phi(x_1)\; \mathrm{tr}\, S_F(0) \;\stackrel{\mathrm{def}}{=}\; \mathrm{i}g\; \mathrm{Tr}\,(\phi\, S_F) \end{aligned} \tag{1.32}$$

in which the symbol [tr] denotes the sum over spinor indices. The next term can be handled in a quite similar way by making repeatedly use of the completeness relation

$$\int \mathrm{d}x\; | x \rangle \langle x | \;=\; \mathbb{I}$$

Then we obtain

$$\begin{aligned} X_\phi^2 &= (-\mathrm{i}g)^2 \int \mathrm{d}x_1 \int \mathrm{d}x_2\; \phi(x_1)\, \phi(x_2) \\ &\times\; (-1)\,\mathrm{tr}\; S_F(x_2 - x_1)\, S_F(x_1 - x_2) \end{aligned} \tag{1.33}$$

It is convenient to introduce the *center of mass* and relative coordinates

$$\bar{x} = \frac{1}{2}(x_1 + x_2) \qquad x = x_1 - x_2 \qquad \frac{\partial(x_1, x_2)}{\partial(X, x)} = 1$$

so that

$$X_\phi^2 \;=\; g^2 \int \mathrm{d}\bar{x} \int \mathrm{d}x\; \phi(\bar{x} + x/2)\, \phi(\bar{x} - x/2)\; \mathrm{tr}\; S_F(-x)\, S_F(x)$$

$$
\begin{aligned}
= \quad & g^2 \int \mathrm{d}\bar{x} \int \mathrm{d}x \int \frac{\mathrm{d}\ell}{(2\pi)^4}\, \widetilde{\phi}(\ell) \int \frac{\mathrm{d}k}{(2\pi)^4}\, \widetilde{\phi}(k) \\
\times \quad & \exp\{-\mathrm{i}\bar{x}\cdot(\ell+k) - \mathrm{i}x\cdot(\ell-k)/2\} \int \frac{\mathrm{d}p}{(2\pi)^4} \int \frac{\mathrm{d}q}{(2\pi)^4} \\
\times \quad & \exp\{\mathrm{i}(p-q)\cdot x\}\ \mathrm{tr}\, \frac{\mathrm{i}}{\not p - M + \mathrm{i}\varepsilon}\ \frac{\mathrm{i}}{\not q - M + \mathrm{i}\varepsilon} \\
= \quad & -\, g^2 \int \frac{\mathrm{d}k}{(2\pi)^4}\, \widetilde{\phi}(k)\, \widetilde{\phi}(-k) \int \frac{\mathrm{d}p}{(2\pi)^4} \int \frac{\mathrm{d}q}{(2\pi)^4} \\
\times \quad & \frac{\mathrm{tr}\,(\not p + M)(\not q + M)}{(p^2 - M^2 + \mathrm{i}\varepsilon)(q^2 - M^2 + \mathrm{i}\varepsilon)}\ (2\pi)^4\, \delta(k + p - q) \\
= \quad & (-\mathrm{i}g)^2 \int \frac{\mathrm{d}k}{(2\pi)^4}\, \widetilde{\phi}(k)\, \widetilde{\phi}(-k) \\
\times \quad & (-1) \int \frac{\mathrm{d}p}{(2\pi)^4}\ \mathrm{tr}\, \frac{\mathrm{i}}{\not p - M + \mathrm{i}\varepsilon} \cdot \frac{\mathrm{i}}{\not p + \not k - M + \mathrm{i}\varepsilon}
\end{aligned}
$$

the very last line corresponding to a fermion loop with two propagators.

- Hence, whenever a fermion loop appears, it always involves a trace operation over the spinor indices as well as a multiplication by a factor (-1), the ultimate reason of which being due to the anticommuting Grassmann-like nature of the fermion fields. This is the Feynman rule for fermion loops.

A little though will convince the reader [1] that the iteration of the above machinery leads to the result

$$
\begin{aligned}
& X_\phi^n = (-\mathrm{i}g)^n \int \frac{\mathrm{d}k_1}{(2\pi)^4}\, \widetilde{\phi}(k_1) \cdots \int \frac{\mathrm{d}k_n}{(2\pi)^4}\, \widetilde{\phi}(k_n)\ (2\pi)^4\, \delta(K) \\
& \times (-1) \int \frac{\mathrm{d}p}{(2\pi)^4}\ \mathrm{tr}\left[\widetilde{S}_F(p)\, S_F(p + k_1) \cdots S_F(p + k_1 + \cdots + k_n)\right] \\
& = (-1)\,(-\mathrm{i}g)^n\ \mathrm{Tr}\,(\phi\, S_F)^n \qquad (K = k_1 + \cdots + k_n) \qquad (1.34)
\end{aligned}
$$

which corresponds to a fermion loop with $n-$external legs associated to the scalar field vertices, as depicted in fig. N 6. As a consequence, we can see by direct inspection that the symbolic equality (1.31) can be understood in a perturbative sense as a power series in the Yukawa coupling, the $n-$th

[1] The trick is to introduce a change of variables to a new system of coordinates with the *center of mass* and $(n-1)$ relative coordinates with a unit Jacobian.

coefficient of which does involve the 1-loop fermion boxes with $n-$external scalar legs with momenta $k_1, k_2, \ldots, k_n$: namely,

$$\frac{\det \| \, \mathrm{i}\,\partial\!\!\!/ - M - g\,\phi \, \|}{\det \| \, \mathrm{i}\,\partial\!\!\!/ - M \, \|} = \exp\left\{ (-1) \sum_{n=1}^{\infty} \frac{1}{n} \, (\,-\,\mathrm{i}\, g\,)^n \,\mathrm{Tr}\,(\,\phi\, S_F\,)^n \right\}$$

or even

$$\det \| \, \mathrm{i}\,\partial\!\!\!/ - M - g\,\phi \, \| = \det \| \, \mathrm{i}\,\partial\!\!\!/ - M \, \| \, \exp\{ \,\mathrm{Tr}\,\ln(\mathbb{I} + ig\,\phi\, S_F\,)\,\}$$

as naïvely expected by taking into account the suggestive symbolic relation $(\,\mathrm{i}\,\partial\!\!\!/ - M\,)^{-1} = -\,\mathrm{i}\, S_F$, as well as $\mathrm{Tr}\,(\,\phi\, S_F\,) = \mathrm{Tr}\,(\,S_F\,\phi\,)$, *i.e.* the cyclicity property of the Tr operation. Notice however that the first two coefficients of the perturbative expansion, that is

$$(-1)\, S_F\,(0) = (-1) \int \frac{\mathrm{d}p}{(2\pi)^4} \;\mathrm{tr}\, \frac{\mathrm{i}}{p\!\!\!/ - M + \mathrm{i}\varepsilon} \qquad \text{for } n = 1$$

$$(-1) \int \frac{\mathrm{d}p}{(2\pi)^4} \;\mathrm{tr}\, \frac{\mathrm{i}}{p\!\!\!/ - M + \mathrm{i}\varepsilon} \; \frac{\mathrm{i}}{p\!\!\!/ + k\!\!\!/ - M + \mathrm{i}\varepsilon} \qquad \text{for } n = 2$$

appear to be ultraviolet divergent. Hence, they call for some of regularization method in order to be properly defined. I shall deal at length with this in the sequel.

1.3.2 Yukawa potential

Let me now consider, for the sake of pedagogical simplicity, the scattering of two distinguishable fermions of mass M , *e.g.* two particles of negative charge $-e$ and $-Ze$ respectively, or two antiparticles of positive charge $+e$ and $+Ze$, in the non-relativistic approximation. In such a circumstance, by comparing the amplitude for this process to the Born approximation formula for non-relativistic quantum mechanics, we can extract the potential $V(r)$ created by the Yukawa field theory model.

As the two colliding fermions are supposed to be distinguishable, only the diagram of fig. N 7 does contribute to the lowest order g^2 . Actually, to be definite, we understand the incoming particles as free spinor particles of given energy-momentum and polarization (p, r) and (q, s) , while the outgoing free particles will carry the energy-momentum and polarization labels (p', r') and (q', s') respectively. Hence, in drawing the Feynman rules for this process, we replace the incoming fermion propagators of momenta p and q by the incident spin states $u_r\,(p)$ and $u_s\,(q)$, whereas the outgoing fermions propagators of momenta p' and q' will be replaced by $\bar{u}_{r'}(p')$ and $\bar{u}_{s'}(q')$ respectively. Of course we have $r, s, r', s' = 1, 2$, $p^2 = q^2 = p'^2 = q'^2 = M^2$.

In the non-relativistic limit we can approximate as follows :

$$\begin{aligned} p &\approx (M, \mathbf{p}) & q &\approx (M, \mathbf{q}) \\ p' &\approx (M, \mathbf{p}') & q' &\approx (M, \mathbf{q}') \\ (p - p')^2 &\approx -|\mathbf{p} - \mathbf{p}'|^2 & u_r(p) &\approx \xi_r \sqrt{M} \quad \text{et cetera} \\ \bar{u}_{r'}(p')\, u_r(p) &\approx 2M\, \delta_{rr'} & \bar{u}_{s'}(q')\, u_s(q) &\approx 2M\, \delta_{ss'} \end{aligned}$$

in such a manner that spin of each particle is conserved in the non-relativistic regime. Putting all pieces together we find the Feynman graph transition amplitude

$$\bar{u}_{r'}(p')\, u_r(p)\, \frac{-\mathrm{i} g^2}{(p - p')^2 - m^2}\, \bar{u}_{s'}(q')\, u_s(q)$$

and in the non-relativistic approximation we are left with

$$\frac{\mathrm{i} g^2}{|\mathbf{p} - \mathbf{p}'|^2 + m^2}\, 2M\, \delta_{rr'}\, 2M\, \delta_{ss'} = \mathrm{i}\, T_{\mathbf{p}, \mathbf{p}'}\, 2M\, \delta_{rr'}\, \delta_{ss'}$$

with $\mathbf{p} + \mathbf{q} = \mathbf{p}' + \mathbf{q}'$, in which I have suitably factorized the non-relativistic center of mass energy $2M$ so that the (relative motion) transition amplitude

$$T_{\mathbf{p}, \mathbf{p}'} = \frac{2M g^2}{|\mathbf{p} - \mathbf{p}'|^2 + m^2}$$

has the dimensions of a length in physical units, its square modulus being the differential cross section.

In non-relativistic quantum mechanics, the scattering amplitude in the Born approximation for a spinless particle in a time independent potential $V(\mathbf{r})$ is given by[2]

$$\begin{aligned} f(\theta) &= -\frac{\mu}{2\pi \hbar^2} \langle \mathbf{p}' | V | \mathbf{p} \rangle \\ &= -\frac{\mu}{2\pi \hbar^2} \int \mathrm{d}\mathbf{r}\ \exp\{-\mathrm{i}(\mathbf{p} - \mathbf{p}') \cdot \mathbf{r}\}\ V(\mathbf{r}) \end{aligned}$$

where μ is the particle mass, θ is the scattering angle for the elastic process with $|\mathbf{p}| = |\mathbf{p}'|$. Hence, the differential cross section for a scattering process by a static fixed target potential is simply provided by

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \right) = |f(\theta)|^2 \qquad \mathrm{d}\Omega = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$$

[2] see *e.g.* Eugene Merzbacher, *Quantum Mechanics*, John Wiley & Sons, New York (1970) §**4** p.229

Consider the attractive Yukawa central potential

$$V(r) = \begin{cases} +\infty & \text{for } 0 \le r < a \\ -\hbar V_0 (4\pi\mu c r)^{-1} \mathrm{e}^{-\mu c r/\hbar} & \text{for } \quad r \ge a \end{cases}$$

in which a is the hard−core repulsive barrier of nucleon impenetrability, while the characteristic Compton wavelength $(\hbar/\mu c)$ may be identified with the range of the potential. Turning back to natural units we get

$$\begin{aligned} f(\theta) &= V_0 \int_{-1}^{1} \mathrm{d}\tau \int_a^\infty r\,\mathrm{d}r\; \exp\{-r(\mu + \mathrm{i}\tau\,|\mathbf{p}-\mathbf{p}'|)\} \\ &= 2V_0 \int_a^\infty \mathrm{d}r\; \frac{\sin(|\mathbf{p}-\mathbf{p}'|\,r)}{|\mathbf{p}-\mathbf{p}'|}\; \mathrm{e}^{-\mu r} \\ &\overset{a\to 0}{\sim} \frac{2V_0}{|\mathbf{p}-\mathbf{p}'|^2 + \mu^2} = \frac{2V_0}{4|\mathbf{p}|^2 \sin^2(\theta/2) + \mu^2} \end{aligned}$$

Now it is apparent that if we identify $\mu = m$, $V_0 = M g^2$ then $f(\theta) = T_{\mathbf{p},\mathbf{p}'}$ and thereby

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right) = \frac{4M^2 g^4}{\left(4|\mathbf{p}|^2 \sin^2(\theta/2) + m^2\right)^2}$$

For a nuclear force range $r \sim 1$ fm we find a meson mass $m \sim 197$ MeV which is not too far from the neutral pion mass $m_{\pi^0} = 134.9766 \pm 0.0006$ MeV, keeping in mind the crudeness of the approximation. Furthermore, from the phenomenological evidence that the nuclear force is 10^3 times the Coulomb force at a distance of 1 fm, we get the order of magnitude of the Yukawa coupling

$$g^2 \sim \frac{\alpha\, m_p}{4\pi\,\mathrm{e}\, m_\pi} \times 10^3 \qquad \Rightarrow \qquad g \sim 1 \div 10$$

1.4 Quantum Electrodynamics

Let me finally come to a fully realistic field theoretic model : the spinor quantum electrodynamics, *i.e.* the Lorentz covariant theory of quantized spinor matter interacting with quantized electromagnetic radiation. The corresponding classical Lagrange density is provided by $(\hbar = 1 = c)$

$$\mathcal{L} = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi} \left(\tfrac{1}{2}\mathrm{i}\overleftrightarrow{\partial\!\!\!/} - M + e A\!\!\!/ \right) \psi \tag{1.35}$$

and turns out to be invariant under the so called *local gauge symmetry*, *i.e.* the space-time point dependent transformations

$$\begin{cases} A_\mu(x) & \mapsto \quad A'_\mu(x) = A_\mu(x) + \partial_\mu f(x) \\ \psi(x) & \mapsto \quad \psi'(x) = \exp\{\mathrm{i}e f(x)\}\,\psi(x) \\ \bar{\psi}(x) & \mapsto \quad \bar{\psi}'(x) = \bar{\psi}(x)\, \exp\{-\mathrm{i}e f(x)\} \end{cases} \tag{1.36}$$

in which the dimensionless coupling parameter $e(\hbar c)^{-\frac{1}{2}}$ appears, $(-e)$ being the negative electron charge, while $f(x)$ is any arbitrary real function. Now, just owing to the gauge symmetry, the kinetic term for the vector potential

$$-\tfrac{1}{4}\,F^{\mu\nu}\,F_{\mu\nu} = \tfrac{1}{2}\,A^{\mu}\,(\,g_{\mu\nu}\,\Box - \partial_{\mu}\,\partial_{\nu}\,)\,A^{\nu} + \text{irrelevant}$$

can not be inverted and thereby a Feynman propagator can not be defined. Hence, in the aim to develop Lorentz covariant perturbation theory, the so called general covariant *gauge fixing Lagrangian* must be added, *viz.*

$$\mathcal{L}_{\text{g.f.}} = A^{\mu}\,(x)\,\partial_{\mu}\,B\,(x)\; + \tfrac{1}{2}\,\xi\,B^{2}\,(x)$$

where $B\,(x)$ is an auxiliary unphysical scalar field of canonical engeneering dimension $[\,B\,] = \text{eV}^{2}$, while the dimensionless parameter $\xi \in \mathbb{R}$ is named the *gauge fixing parameter*, the abelian field strength being as usual $F_{\mu\nu}\,(x) = \partial_{\mu}\,A_{\nu}\,(x) - \partial_{\nu}\,A_{\mu}\,(x)$, in such a manner that the action results to be Poincaré invariant. If we definitely make the simplest choice $\xi = 1$, that is called the Feynman gauge, after turning to the momentum space we find

$$\begin{aligned}
\mathrm{i}\,S\,[\,A^{\mu},\psi,\bar{\psi}\,] \;&=\; -\,\frac{\mathrm{i}}{2}\,g_{\mu\nu}\int\frac{\mathrm{d}k}{(\,2\pi\,)^{4}}\;\widetilde{A}^{\mu}\,(k)\,k^{2}\,\widetilde{A}^{\nu}\,(-k)\\
&+\;\mathrm{i}\int\frac{\mathrm{d}p}{(\,2\pi\,)^{4}}\;\widetilde{\psi}^{\,\dagger}\,(p)\,\gamma^{0}\,(\,\not{p} - M\,)\;\widetilde{\psi}\,(p)\\
&+\;\mathrm{i}\,e\int\frac{\mathrm{d}k}{(\,2\pi\,)^{4}}\int\frac{\mathrm{d}p}{(\,2\pi\,)^{4}}\int\frac{\mathrm{d}q}{(\,2\pi\,)^{4}}\;\widetilde{A}_{\mu}\,(k)\,\widetilde{\psi}^{\,\dagger}\,(q)\,\gamma^{0}\,\gamma^{\mu}\,\widetilde{\psi}\,(p)\\
&\times\;(\,2\pi\,)^{4}\,\delta\,(\,k + p - q\,)
\end{aligned}$$

in such a manner that, in close resemblance with the Yukawa theory, we immediately come to the Feynman rules

$$\text{vertex}:\qquad \mathrm{i}\,e\gamma^{\mu} \tag{1.37}$$

$$\text{photon propagator}:\qquad -\,g^{\mu\nu}\,\frac{\mathrm{i}}{k^{2} + \mathrm{i}\varepsilon} \tag{1.38}$$

$$\text{spinor propagator}:\qquad \frac{\mathrm{i}\,(\,\not{p} + M\,)}{p^{2} - M^{2} + \mathrm{i}\varepsilon} \tag{1.39}$$

while the energy-momentum conservation is again $(\,2\pi\,)^{4}\,\delta\,(\,k + p - q\,)$ and a factor (-1) must be included for each fermion loop. As a simple application of these Feynman rules, let me repeat the nonrelativistic scattering amplitude of the previous section, this time for quantum electrodynamics. For two incoming and two outgoing particles of equal masses but unlike charges $-e$ and $-Ze$ respectively, the leading order contribution is

$$\bar{u}_{r'}\,(p')\,(\,\mathrm{i}\,e\gamma^{\,\mu}\,)\,u_{r}\,(p)\;\frac{-\,\mathrm{i}}{(p - p')^{2}}\;\bar{u}_{s'}\,(q')\,(\,\mathrm{i}\,Ze\gamma_{\,\mu}\,)\,u_{s}\,(q) \tag{1.40}$$

Once again, in the nonrelativistic limit

$$\bar{u}_{r'}(p')\,\gamma^0\,u_r(p) \;\approx\; 2M\,\delta_{rr'} \qquad \text{et cetera}$$

where M is the particle common mass in such a manner that we can write

$$\frac{-\,\mathrm{i}\,Ze^2}{|\,\mathbf{p}-\mathbf{p}'\,|^2}\;2M\,\delta_{rr'}\;2M\,\delta_{ss'} = \mathrm{i}\,T_{\mathbf{p},\mathbf{p}'}\;2M\,\delta_{rr'}\,\delta_{ss'}$$

and consequently

$$T_{\mathbf{p},\mathbf{p}'} = f(\theta) = -\,\frac{2MZe^2}{|\,\mathbf{p}-\mathbf{p}'\,|^2}$$

which corresponds to the repulsive Coulomb potential

$$V(r) = \frac{Ze^2}{4\pi r} = Z\,\frac{\alpha}{r}$$

so that

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right) = \frac{M^2Z^2\alpha^2}{4\,|\,\mathbf{p}\,|^4\,\sin^4(\theta/2)} = \frac{Z^2\alpha^2}{16E^2\,\sin^4(\theta/2)} \qquad (\,\mathbf{p}^2 = 2ME\,)$$

which is nothing but the celebrated Rutherford exact cross section. Notice that for particle−antiparticle scattering, owing to

$$\bar{v}_{r'}(p')\,\gamma^0\,v_r(p) \;\approx\; -\,2M\,\delta_{rr'} \qquad \text{et cetera}$$

the sign of the non-relativistic Coulomb potential is opposite as it does.

As a final important remark, I start remembering the Euler-Lagrange field equations, that hold true in the Feynman gauge, for the operator valued tempered distributions $A^\mu(x)$, $B(x)$ and $\psi(x)$: namely,

$$\Box A^\mu(x) = \jmath^\mu(x) \tag{1.41}$$

$$\mathrm{i}\,\partial\!\!\!/\,\psi(x) + e\,A\!\!\!/(x)\,\psi(x) = M\,\psi(x) \tag{1.42}$$

$$\partial_\mu A^\mu(x) = B(x) \tag{1.43}$$

where the local electric current quantum operator is defined as

$$\jmath^\mu(x) = (-e)\lim_{\epsilon\to 0}\,\tfrac{1}{2}\Big(\bar{\psi}(x+\epsilon)\,\gamma^\mu\,\psi(x) - \bar{\psi}(x)\,\gamma^\mu\,\psi(x+\epsilon)\Big)$$

in such a manner to avoid the ill-defined product of tempered distribution at the same space-time point, the relative minus sign being due to the canonical equal time anticommutation relations

$$\{\,\psi(t,\mathbf{x})\,,\,\psi^\dagger(t,\mathbf{y})\,\} \;=\; \delta\,(\mathbf{x}-\mathbf{y})$$

all the other anticommutators vanishing. From the 4-divergence of equation (1.41) and taking equation (1.43) into account we find

$$\Box B(x) = \partial_\mu \jmath^\mu(x)$$

On the other side, from the invariance of the Lagrangian with respect to the $U(1)$ global, *i.e.* space-time point independent, phase transformations

$$\begin{cases} \psi(x) & \mapsto \quad \psi'(x) = \exp\{\mathrm{i}e\theta\}\,\psi(x) \\ \bar\psi(x) & \mapsto \quad \bar\psi'(x) = \bar\psi(x)\,\exp\{-\mathrm{i}e\theta\} \end{cases}$$

it follows that the electric current density four vector satisfies the continuity equation and the total electric charge is conserved, which is nothing but the Noether's theorem. Hence

$$\partial_\mu \jmath^\mu(x) \equiv 0 \qquad \Leftrightarrow \qquad \Box B(x) = 0$$

which means that the auxiliary scalar $B(x)$ is still a free field obeying the d'Alembert wave equation *even in the presence of the interaction.* It follows that it is still possible to select the Hilbert space of the physical states, with a positive semidefinite norm, from the subsidiary condition

$$B^{(-)}(x)\,|\,\mathrm{phys}\rangle = 0$$

just like in the Gupta-Bleuler or Nakanishi-Lautrup formalism for the free radiation quantum field.

1.5 Euclidean field theories

The Feynman's rules for euclidean field theories can be readily obtained in accordance with the main guidelines I have discussed in the framework of the quantum field theories in the Minkowski space−time. To be definite, let me consider the Yukawa−self−interacting scalar−fermion euclidean model, which is descibed by the euclidean action

$$\begin{aligned} S_E[\,\phi_E\,,\,\psi_E\,,\,\bar\psi_E\,] \;&=\; \int \mathrm{d}x_E\;\left(\tfrac12\,\partial_\mu\phi_E\,\partial_\mu\phi_E + \tfrac12\,m^2\,\phi_E^2 + \tfrac{\lambda}{4!}\,\phi_E^4\right) \\ &+\; \int \mathrm{d}x_E\;\bar\psi_E\,(\,\partial\!\!\!/_E + M + g\,\phi_E\,)\,\psi_E \end{aligned} \tag{1.44}$$

where as usual

$$x_{E\mu} = (\,\mathbf{x}\,,\,x_4 = -\,i\,x^0\,) \qquad \partial_\mu = \frac{\partial}{\partial x_{E\mu}} \qquad x_{E\mu}\,y_{E\mu} = \mathbf{x}\cdot\mathbf{y} + x_4\,y_4$$

$$\mathrm{i}\,\partial\!\!\!/_E = \bar{\gamma}_\mu \frac{\partial}{\partial x_{E\mu}} \qquad \left\{\bar{\gamma}_\mu\,,\,\bar{\gamma}_\nu\right\} = 2\,\delta_{\mu\nu} \qquad \bar{\gamma}_\mu = \bar{\gamma}_\mu^{\,\dagger}$$

The generating functional for euclidean disconnected correlation functions will be defined by

$$\begin{aligned} Z_E[\,J_E\,,\,\bar{\zeta}_E\,,\,\zeta_E\,] &= \exp\left\{-W_E[\,J_E\,,\,\zeta_E\,,\,\bar{\zeta}_E\,]\right\} \\ &= \exp\left\{-V_E[\,\delta/\delta J_E\,,\,\delta/\delta\zeta_E\,,\,\delta/\delta\bar{\zeta}_E]\right\} \\ &\times\ Z_E^0[\,J_E\,,\,\bar{\zeta}_E\,,\,\zeta_E\,] \\ &\stackrel{\mathrm{def}}{=} \exp\left\{-V_E[\,\delta/\delta J_E\,,\,\delta/\delta\zeta_E\,,\,\delta/\delta\bar{\zeta}_E]\right\} \\ &\times\ \exp\left\{-W_E^0[\,J_E\,,\,\zeta_E\,,\,\bar{\zeta}_E\,]\right\} \end{aligned} \tag{1.45}$$

where

$$\begin{aligned} Z_E^0[\,J_E\,,\,\bar{\zeta}_E\,,\,\zeta_E\,] &= \mathcal{N}\int \mathfrak{D}\phi_E\,\mathfrak{D}\psi_E\,\mathfrak{D}\bar{\psi}_E\ \exp\left\{-S_E^0[\,\phi_E\,,\,\psi_E\,,\,\bar{\psi}_E\,]\right\} \\ &\times\ \exp\int \mathrm{d}x_E\ \left(\phi_E\,J_E + \bar{\psi}_E\,\zeta_E + \bar{\zeta}_E\,\psi_E\right) \end{aligned} \tag{1.46}$$

$$\begin{aligned} &S_E^0[\,\phi_E\,,\,\psi_E\,,\,\bar{\psi}_E\,] = \\ &\int \mathrm{d}x_E\ \left[\,\tfrac{1}{2}\,\partial_\mu\phi_E\,\partial_\mu\phi_E + \tfrac{1}{2}\,m^2\,\phi_E^2 + \bar{\psi}_E\,(\,\partial\!\!\!/_E + M\,)\,\psi_E\,\right] \end{aligned} \tag{1.47}$$

$$W_E^0[\,J_E\,,\,\zeta_E\,,\,\bar{\zeta}_E\,] = \left\langle\,\tfrac{1}{2}\,J_{Ex}\,D_{xy}^E\,J_{Ey} + \bar{\zeta}_{Ex}\,S_{xy}^E\,\zeta_{Ey}\,\right\rangle$$

in which

$$D_E(x_E) = \frac{1}{(2\pi)^4}\int \mathrm{d}k_E\ \frac{\exp\{\mathrm{i}\,k_E\cdot x_E\}}{k_E^2 + m^2}$$

$$S_{\alpha\beta}^E(x_E) = \int \frac{\mathrm{d}k_E}{(2\pi)^4}\ \exp\{\mathrm{i}\,k_E\cdot x_E\}\ \left(\frac{\mathrm{i}}{-\,p\!\!\!/_E + \mathrm{i}M}\right)_{\alpha\beta}$$

Then the Feynman's rules immediately follows and read

euclidean scalar propagator : $\dfrac{1}{k_E^2 + m^2}$

euclidean spinor propagator : $\dfrac{\mathrm{i}}{-\,p\!\!\!/_E + \mathrm{i}M} = \dfrac{-\,\mathrm{i}\,p\!\!\!/_E + M}{p_E^2 + M^2}$

euclidean scalar vertex : $-\dfrac{\lambda}{4!}$ $\quad\left(\bar{k}_1 + \bar{k}_2 + \bar{k}_3 + \bar{k}_4\ =\ 0\right)$

euclidean Yukawa vertex : $-\,g$ $\quad\left(\bar{p} + \bar{q} + \bar{k}\ =\ 0\right)$

where all momenta are supposed to be incoming [3] while $k_E \equiv \bar{k}$ and $p_E \equiv \bar{p}$

$$\text{euclidean scalar loop :} \qquad \int \frac{\mathrm{d}^4\bar{k}}{(2\pi)^4}$$

$$\text{euclidean spinor loop :} \qquad (-1)\int \frac{\mathrm{d}^4\bar{p}}{(2\pi)^4}$$

the symmetry factors being obviously the same as in Minkowski space−time. The euclidean formulation of quantum electrodynamics is also achieved after a straightforward generalization of the above recipe. To be specific we have

$$\bar{A}_\mu = (\,\mathbf{A}\,,\,A_4 = -\,\mathrm{i}\,A_0\,) \qquad \bar{F}_{\mu\nu} = \partial_\mu\,\bar{A}_\nu - \partial_\nu\,\bar{A}_\mu \tag{1.48}$$

$$\begin{aligned} S_E[\,\bar{A}_\mu\,,\,B_E\,,\,\bar{\psi}_E\,,\,\psi_E\,] \;&=\; \int \mathrm{d}\bar{x}\,\Big\{\tfrac{1}{4}\,\bar{F}_{\mu\nu}\,\bar{F}_{\mu\nu} - B_E\,\partial_\mu\,\bar{A}_\mu + \tfrac{1}{2}\,B_E^2 \\ &+\; \bar{\psi}_E\left(\partial\!\!\!/_E + M - e\;\bar{\gamma}_\mu\,\bar{A}_\mu\right)\psi_E\Big\} \end{aligned}$$

and thereby

$$\text{euclidean photon propagator :} \qquad \frac{\delta_{\mu\nu}}{k_E^2 + m^2}$$

$$\text{euclidean photon} - \text{spinor vertex :} \qquad e\;\bar{\gamma}_\mu \qquad (p_E + k_E + q_E \;=\; 0\,)$$

The connected euclidean Green's functions are named correlation functions or Schwinger's functions and are given by

$$G_E^{(n)}\,(\bar{x}_1\,,\,\ldots\,,\,\bar{x}_n\,) \;=\; -\;\delta^{(n)}\,W_E[\,J_E\,]\,/\,\delta\,J_E(\bar{x}_1)\,\cdots\,\delta\,J_E(\bar{x}_n)\,\big\rfloor_{J_E=0} \tag{1.49}$$

Now it is very instructive to compare the Feynman rules for quantum field theories in the Minkowski space−time and the corresponding euclidean counteparts. In the case of the self−interacting scalar field we find

$$\frac{1}{k_E^2 + m^2} \qquad \text{propagator} \qquad \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon}$$

$$-\;\frac{\lambda}{4!} \qquad \text{vertex} \qquad -\;\frac{\mathrm{i}\,\lambda}{4!}$$

$$\int \frac{\mathrm{d}^4\,\ell_E}{(2\pi)^4} \qquad \text{loop integration} \qquad \int \frac{\mathrm{d}^4\,\ell}{(2\pi)^4}$$

[3] We have to remember that ψ_E and $\bar{\psi}_E$ are truly independent.

while for Yukawa neutral meson theory and quantum electrodynamics

$$\frac{\mathrm{i}}{-\not{p}_E + \mathrm{i}M} \qquad \text{propagator} \qquad \frac{\mathrm{i}}{\not{p} - M + \mathrm{i}\varepsilon}$$

$$\frac{\delta_{\mu\nu}}{k_E^2 + m^2} \qquad \text{propagator} \qquad \frac{-\,\mathrm{i}\,g_{\mu\nu}}{k^2 - m^2 + \mathrm{i}\varepsilon}$$

$$-\;g\,\delta\,(p_E + k_E + q_E) \qquad \text{vertex} \qquad -\;\mathrm{i}g\,\delta\,(p + k - q)$$

$$e\;\bar{\gamma}_\mu\,\delta\,(p_E + k_E + q_E) \qquad \text{vertex} \qquad \mathrm{i}\,e\,\gamma^\mu\,\delta\,(p + k - q)$$

$$(-1)\int\frac{\mathrm{d}^4 p_E}{(2\pi)^4} \qquad \text{spinor loop integration} \qquad (-1)\int\frac{\mathrm{d}^4 p}{(2\pi)^4}$$

As a consequence, the transition from a connected $n-$point Schwinger's function to a connected $n-$point Green's function in momentum space can be readily achieved. We shall se how to proceed in the forthcoming sections.

Chapter 2

Scattering matrix

2.1 The S-matrix in quantum mechanics

Consider an isolated mechanical system with a time-independent self-adjoint hamiltonian operator

$$H = H_0 + V$$

acting upon the Hilbert space $\mathfrak{H}$ of the system, and let $\psi \in \mathfrak{H}$ a proper state of this quantum mechanical system. Without loss of generality, the free hamiltonian operator H_0 is supposed to be self-adjoint, time-independent and endowed with a purely continuous spectrum. Moreover, the interaction potential is supposed to fall down to zero at large distances according to

$$V(\mathbf{r}) \stackrel{r\to\infty}{\sim} O\left(r^{-3/2-\epsilon}\right) \qquad (r = |\mathbf{r}| \quad \epsilon > 0)$$

which corresponds to sufficiently short-range interactions. Then, the proper *asymptotic states* $\psi_{\rm as}$ exist, which are related to the proper states $\psi \in \mathfrak{H}$, characterized by the following behaviour :

$$\exp\{-\,\mathrm{i}\,H_0 t\}\ |\psi\rangle \stackrel{|t|\to\infty}{\sim} \exp\{-\,\mathrm{i}\,H t\}\ |\psi_{\rm as}\rangle$$

In other terms, the asymptotic in- and out-states are defined by

$$\mathrm{s}-\lim_{t\to\mp\infty}\ \exp\{\,\mathrm{i}\,H t\}\ \exp\{-\,\mathrm{i}\,H_0 t\}\ |\psi\rangle \ \stackrel{\rm def}{=} \begin{cases} |\psi_{\rm in}\rangle & (t \to -\infty) \\ |\psi_{\rm out}\rangle & (t \to +\infty) \end{cases}$$

where the symbol $\mathrm{s}-\lim$ stands for the limit in the strong topology of the Hilbert space $\mathfrak{H}$, that is

$$\|\ \exp\{\,\mathrm{i}\,H t\}\ \exp\{-\,\mathrm{i}\,H_0 t\}\,\psi - \psi_{\rm as}\ \| \to\ 0 \qquad |\,t\,| \to \infty$$

In such a manner, we can define the isometric asymptotic wave-operators

$$\Omega_{\rm as} \stackrel{\rm def}{=} {\rm s} - \lim_{\,|\,t\,|\,\to\,\infty} \ \exp\{\,{\rm i}\,H\,t\}\ \exp\{-\,{\rm i}\,H_0\,t\} \tag{2.1}$$

where

$$\Omega_{\rm as} \ : \ \mathfrak{H} \ \longrightarrow \ \mathfrak{H}_{\rm as} \ \equiv \ \mathfrak{H} \setminus \mathfrak{B}$$

in which the subset $\mathfrak{B} \subset \mathfrak{H}$ of the Hilbert space is spanned by the bound states of the hamiltonian operator H, *i.e.* the eigenstates belonging to the purely discrete part of the spectrum of H. Hence, in general, the asymptotic wave-operators are not unitary but only isometric because

$$\| \ \psi_{\rm as} \ \| = \| \ \Omega_{\rm as}\,\psi \ \| = \| \ \psi \ \|$$

$$\Omega_{\rm as}\,|\,\psi\,\rangle = |\,\psi_{\rm as}\,\rangle \in \mathfrak{H} \setminus \mathfrak{B}$$

Notice that, according to the fundamental theorem for self-adjoint operators, the complementary subspaces $\mathfrak{B}$ and $\mathfrak{H} \setminus \mathfrak{B}$ are mutually orthogonal.

The $S-$matrix is a unitary operator $S \ : \ \mathfrak{H} \ \longrightarrow \ \mathfrak{H}$ and is defined by

$$\begin{aligned} S \ &\stackrel{\rm def}{=} \ \Omega^{\dagger}_{\rm out}\,\Omega_{\rm in} \\ &= \ \lim_{t\,\to\,+\infty}\ \lim_{t'\,\to\,-\infty}\ \exp\{\,{\rm i}\,H_0\,t\}\ \exp\{-\,{\rm i}\,H\,(t-t')\}\ \exp\{-\,{\rm i}\,H_0\,t'\} \\ &= \ \lim_{t\,\to\,+\infty}\ \lim_{t'\,\to\,-\infty}\ \exp\{\,{\rm i}\,H_0\,t\}\ U\,(t\,,\,t')\ \exp\{-\,{\rm i}\,H_0\,t'\} \\ &= \ \lim_{t\,\to\,+\infty}\ U_{\rm int}\,(t\,,\,-t) \end{aligned} \tag{2.2}$$

where the limits are now understood in the weak topology of the Hilbert space, that is

$$(\,\varphi\,,\,S\,\psi\,) = \langle\,\varphi_{\rm out}\,|\,\psi_{\rm in}\,\rangle \qquad\qquad \forall\,\varphi\,,\,\psi \in \mathfrak{H}$$

while

$$U_{\rm int}\,(t\,,\,-t) = \exp\{\,{\rm i}\,H_0\,t\}\ U\,(t\,,\,-\,t)\ \exp\{\,{\rm i}\,H_0\,t\}$$

is the evolution operator in the interaction picture. In the interaction picture the time evolution of operators is governed by the free part H_0 of the complete self-adjoint hamiltonian operator $H = H_0 + V$ of the quantum mechanical system, *i.e.*

$$A_{\rm int}\,(t) = \exp\{\,{\rm i}\,H_0\,t\}\ A\,(0)\ \exp\{\,-\,{\rm i}\,H_0\,t\}$$

where A is any linear operator acting on the Hilbert space $\mathfrak{H}$, while the state vectors obey the equation

$${\rm i}\hbar\,\partial_t\,|\,\psi_{\rm int}\,(t)\,\rangle = V_{\rm int}\,(t)\ |\,\psi_{\rm int}\,(t)\,\rangle$$

with the formal solution

$$|\,\psi_{\rm int}\,(t)\,\rangle = U_{\rm int}\,(\,t\,,\,t'\,)\;|\,\psi_{\rm int}\,(t'\,)\,\rangle$$

$$U_{\rm int}\,(\,t\,,\,t'\,) = T\;\exp\left\{-\,\frac{\rm i}{\hbar}\int_{t'}^{t} d\tau\;V_{\rm int}\,(\tau)\right\}$$

Hence, the scattering operator can be written in the very suggestive form

$$S = U_{\rm int}\,(\,\infty\,,\,-\,\infty\,) = T\;\exp\left\{-\,\frac{\rm i}{\hbar}\int_{-\infty}^{\infty} dt\;V_{\rm int}\,(t)\right\} \tag{2.3}$$

which apparently maps the incoming states from $t\;\to\;-\,\infty$ into outgoing states at $t\;\to\;+\,\infty$, as naïvely expected.

2.2 Green's functions and $S-$matrix

Let me consider, for the sake of simplicity but without loss of generality, the simplest interacting quantum field theory model, *i.e.* , the self-interacting real scalar field model described by the classical Lagrange density

$$\mathcal{L} = \tfrac12\,g^{\,\mu\nu}\,\partial_\mu\,\phi\,(x)\,\partial_\nu\,\phi\,(x) - \tfrac12\,m^2\,\phi^{\,2}\,(x) - \frac{\lambda}{4!}\,\phi^{\,4}\,(x)$$

leading to the conjugate momentum field

$$\Pi\,(x) = \dot{\phi}\,(x)$$

and to the classical hamiltonian functional

$$\begin{aligned} H &= H_0 + V \geq 0 \\ H_0 &= \int {\rm d}\mathbf{x}\;\tfrac12\left[\,\Pi^{\,2}\,(x) - \phi(x)\boldsymbol{\nabla}^{\,2}\phi(x) + m^2\,\phi^{\,2}(x)\,\right] \\ V\,[\,\phi\,] &= \int {\rm d}\mathbf{x}\;\frac{\lambda}{4!}\;\phi^{\,4}(x) \qquad\qquad (\,\lambda > 0\,) \end{aligned}$$

To the aim of understanding the meaning of the interaction representation in the quantum field theory of the interacting fields, let me begin with the real scalar self-interacting quantum field at a given time, *viz.* $x^0 = t = 0$:

$$\phi\,(\,0\,,\,\mathbf{x}\,)\;=\;\frac{1}{(2\pi)^3}\int\frac{{\rm d}\,\mathbf{k}}{2\,\omega_{\mathbf{k}}}\;\left[\,a\,(\mathbf{k})\,{\rm e}^{\,{\rm i}\,\mathbf{k}\cdot\mathbf{x}}\;+\;a^\dagger(\mathbf{k})\,{\rm e}^{-\,{\rm i}\,\mathbf{k}\cdot\mathbf{x}}\,\right]\;\equiv\;\phi\,(\,\mathbf{x}\,) \tag{2.4}$$

where

$$[\,a\,(\mathbf{k})\,,\,a^\dagger(\mathbf{k}'\,)\,] = (2\pi)^3\;2\,\omega_{\mathbf{k}}\;\delta\,(\,\mathbf{k} - \mathbf{k}'\,)$$

all the other commutators being equal to zero. Next I can introduce the field conjugate momentum in a similar way, by means of the equality

$$\Pi(0\,,\,\mathbf{x}) \;=\; \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}\mathbf{k}}{2\mathrm{i}}\,\left[a\,(\mathbf{k})\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \;-\; a^{\dagger}(\mathbf{k})\,\mathrm{e}^{-\,\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\right] \;\equiv\; \Pi\,(\mathbf{x})$$

in such a manner to satisfy the canonical commutation relations

$$[\,\phi\,(\mathbf{x})\,,\,\Pi\,(\mathbf{y})\,] \;=\; \mathrm{i}\,\delta\,(\mathbf{x}-\mathbf{y})$$

all the other commutators vanishing. Then we can easily build up the free hamiltonian quantum operator at $x^0 = t = 0$ that reads

$$H_0 = \int \mathrm{d}\mathbf{x}\ :\tfrac{1}{2}\left[\Pi^{\,2}\,(\mathbf{x}) - \phi\,(\mathbf{x})\boldsymbol{\nabla}^{\,2}\phi\,(\mathbf{x}) + m^2\,\phi^2(\mathbf{x})\right]:$$

where the normal ordering means here that, when I substitute the normal mode expansions for $\Pi\,(\,\mathbf{x}\,)$ and $\phi\,(\,\mathbf{x}\,)$ in the quadratic expression H_0, the creation operators $a^{\dagger}(\mathbf{k})$ stand always to the left of the destruction operators $a\,(\mathbf{k})$, in such a manner that $H_0\,|\,0\,\rangle = 0$. The perturbative $(\lambda = 0)$ vacuum state at $x^0 = t = 0$ is defined by $a\,(\mathbf{k})\,|\,0\,\rangle = 0 = \langle\,0\,|\,a^{\dagger}(\mathbf{k})\ (\forall\,\mathbf{k}\in\mathbb{R}^3\,)$.

Hence we can define the real scalar self-interacting quantum field in the interaction representation by the evolution law

$$\phi_{\,\mathrm{int}}\,(x) \;=\; \mathrm{e}^{\,\mathrm{i}H_0 t}\,\phi\,(\mathbf{x})\,\mathrm{e}^{-\,\mathrm{i}H_0 t} \tag{2.5}$$

which entails the free field theory relationships

$$\Pi_{\,\mathrm{int}}\,(\,t\,,\,\mathbf{x}) = \dot{\phi}_{\,\mathrm{int}}\,(x) = \frac{1}{\mathrm{i}\hbar}\,[\,\phi_{\,\mathrm{int}}\,(\,t\,,\,\mathbf{x}\,)\,,\,H_0\,] \tag{2.6}$$

From the canonical commutation relation, it follows that the real scalar self-interacting quantum field operator in the interaction representation fulfils the Klein-Gordon equation

$$\left(\Box + m^2\right)\phi_{\,\mathrm{int}}\,(x) = 0$$

so that we can write at any time $x^0 = t$ its normal mode decomposition

$$\phi_{\,\mathrm{int}}\,(x) = \int Dk\,\left[\,a\,(k)\,e^{-\,\mathrm{i}\,k\,x} + a^{\dagger}\,(k)\,e^{\,\mathrm{i}\,k\,x}\right] = \phi^{(-)}_{\,\mathrm{int}}\,(x) + \phi^{(+)}_{\,\mathrm{int}}\,(x)$$

$$\int Dk \;\stackrel{\mathrm{def}}{=}\; \int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3\,2\,\omega_{\mathbf{k}}} = \int\frac{\mathrm{d}k}{(2\pi)^3}\;\theta\,(k_0)\,\delta\left(k^2 - m^2\right)$$

$$k^0 \;=\; \omega_{\mathbf{k}} \;=\; \sqrt{\mathbf{k}^2 + m^2}$$
$$\left[\, a\,(k)\,,\, a^\dagger\,(k')\,\right] = (2\pi)^3\, 2\omega_{\mathbf{k}}\, \delta\,(\,\mathbf{k} - \mathbf{k}'\,)$$
$$[\, a\,(k)\,,\, a\,(k')\,] = 0 = \left[\, a^\dagger\,(k)\,,\, a^\dagger\,(k')\,\right]$$

In so arguing, the transition and the correspondence from the non-relativistic quantum mechanics to the relativistic quantum field theory seem to be quite natural and straightforward. Actually, a deeper inspection neatly shows that things are far more complicated, see the comments at the end of this section.

We can now use the suggestive relationship (2.3) in order to express the scattering operator for the self-interacting real scalar field model in terms of the creation and annihilation operators and, moreover, calculate its matrix elements between states containing scalar field quanta of sharply definite energy-momentum. To the aim of calculating those matrix elements it is convenient to deal with normal ordered products, in the interaction picture, with all the creation parts $\phi^{(+)}_{\rm int}\,(x)$ of the scalar field operators standing to the left of the destruction parts $\phi^{(-)}_{\rm int}\,(x)$. This can be done by means of the Wick's theorem

Giancarlo Wick (Torino, 15 ottobre 1909 – Torino, 20 aprile 1992)
Evaluation of the collision matrix
Physical Review **80** (1950) 268

When applied to the chronologically ordered product of n field operators, then Wick's theorem takes the well-known form

$$\begin{aligned} T\,\phi_{\rm int}\,(x_1)\ldots\phi_{\rm int}\,(x_n) \;&=\; :\phi_{\rm int}\,(x_1)\ldots\phi_{\rm int}\,(x_n): \\ &+ \sum_{1\le\imath<\jmath\le n} D_F\,(\,x_\imath - x_\jmath\,)\;:\prod_{\kappa\ne\imath,\jmath}\phi_{\rm int}\,(x_\kappa):\;+\;\cdots \\ + \sum_{1\le\imath_1<\jmath_1\le n}\cdots &\sum_{1\le\imath_r<\jmath_r\le n} D_F\,(\,x_{\imath_1} - x_{\jmath_1}\,)\cdots D_F\,(\,x_{\imath_r} - x_{\jmath_r}\,) \\ &\times \begin{cases} 1 & \text{for}\quad n = 2\ell \qquad\qquad (\,\ell\in\mathbb{N}\,) \\ \phi_{\rm int}\,(x_s) & \text{for}\quad n = 2\ell+1 \quad (\,s\ne\imath_1\ne\cdots\ne\jmath_r\,) \end{cases} \end{aligned} \tag{2.7}$$

$$\left(\Box + m^2\right)\,D_F\,(\,x\,) \;=\; -\,\mathrm{i}\,\delta\,(x) \qquad \text{causal 2 -- point Green's function}$$

The above admittedly rather cumbersone formula can be readily checked by direct inspection for $n = 2$ and can be proved by induction in the general case. For a very detailed and exhaustive proof, which also includes the cases of the spinor and vector fields, see the classic textbook : N.N. Bogoliubov and D.V. Shirkov, *Introduction to the Theory of Quantized Fields*, Interscience Publishers, New York, 1959, §16.2 pp. 159–169, §19.2 pp. 233–235.

A much more compact and convenient functional expression is

$$T\,\mathrm{F}[\phi_{\,\mathrm{int}}] = \exp\left\{\tfrac{1}{2}\int \mathrm{d}x\mathrm{d}y D_F(x-y)\Big(\delta^{(2)}/\delta\phi_{\,\mathrm{int}}(x)\delta\phi_{\,\mathrm{int}}(y)\Big)\right\} : \mathrm{F}[\phi_{\,\mathrm{int}}] :$$

where $\mathrm{F}\,[\,\phi_{\,\mathrm{int}}\,]$ is any functional of the real scalar field in the interacting representation. In particular, if we take the functional

$$\mathrm{F}\,[\,\phi_{\,\mathrm{int}}\,] = \exp\left\{\,\mathrm{i}\int \mathrm{d}x\ \phi_{\,\mathrm{int}}\,(x)\,J\,(x)\,\right\}$$

$J\,(x)$ being as usual some $c-$number external source, we find

$$\begin{aligned} &T\,\exp\left\{\,\mathrm{i}\int \mathrm{d}x\ \phi_{\,\mathrm{int}}\,(x)\,J\,(x)\,\right\} \ = \\ &:\exp\left\{\,\mathrm{i}\int \mathrm{d}x\ \phi_{\,\mathrm{int}}\,(x)\,J\,(x)\,\right\}:\ Z_0\,[\,J\,] \end{aligned} \tag{2.8}$$

where $Z_0\,[\,J\,]$ is the previously introduced generating functional of the Green's functions for the free field theory. It is worthwhile to observe that we can write the functional identity

$$J\,(x)\,Z_0\,[\,J\,] = \mathcal{K}_x\ (\,\delta\,/\,\mathrm{i}\,\delta\,J\,(x)\,)\ Z_0\,[\,J\,] \tag{2.9}$$

in which I have introduced the Klein-Gordon differential operator

$$\mathcal{K}_x \ \stackrel{\mathrm{def}}{=}\ \left(\Box_x + m^2\right)$$

Hence, we can rewrite the functional relationship (2.8) in the form

$$\begin{aligned} &T\,\exp\left\{\,\mathrm{i}\int \mathrm{d}x\ \phi_{\,\mathrm{int}}\,(x)\,J\,(x)\,\right\} \ = \\ &:\exp\left\{\,\mathrm{i}\int \mathrm{d}x\ \phi_{\,\mathrm{int}}\,(x)\,\mathcal{K}_x\ (\,\delta\,/\,\mathrm{i}\,\delta\,J_x\,)\,\right\}:\ Z_0\,[\,J\,] \end{aligned} \tag{2.10}$$

Now we can use once again the expedient which led us to the Feynman rules : namely,

$$\begin{aligned} &T\,\exp\left\{\,-\,\frac{\mathrm{i}\lambda}{4!}\int_{-\infty}^{\infty}\mathrm{d}t\int \mathrm{d}\mathbf{x}\ \phi_{\,\mathrm{int}}^{4}\,(\,t,\mathbf{x}\,)\,\right\} \ = \\ &\exp\left\{\,-\,\mathrm{i}\int \mathrm{d}x\ V\,[\,\delta\,/\,\mathrm{i}\,\delta\,J_x\,]\,\right\}\ T\,\exp\left\{\,\mathrm{i}\int \mathrm{d}y\ \phi_{\,\mathrm{int}}\,(y)\,J\,(y)\,\right\}\Bigg|_{J=0} \end{aligned} \tag{2.11}$$

As a consequence, we can eventually express the scattering operator (2.3) in the rather suggestive form

$$\begin{aligned} S &= U_{\rm int}\,(\infty\,,\,-\infty\,) = T\,\exp\left\{-\frac{\mathrm{i}\lambda}{4!}\int_{-\infty}^{\infty}\mathrm{d}t\int\mathrm{d}\mathbf{x}\;\phi_{\rm int}^{4}\,(t,\mathbf{x}\,)\right\}\\ &\exp\left\{-\,\mathrm{i}\int\mathrm{d}z\;V\,[\,\delta\,/\,\mathrm{i}\,\delta\,J_z\,]\right\}\;T\,\exp\left\{\mathrm{i}\int\mathrm{d}w\;\phi_{\rm int}\,(w)\,J\,(w)\right\}\Bigg|_{J=0}\\ &= \;:\exp\left\{\mathrm{i}\int\mathrm{d}x\;\phi_{\rm int}\,(x)\,\mathcal{K}_x\;(\,\delta\,/\,\mathrm{i}\,\delta\,J_x\,)\right\}:\\ &\times\;\;\exp\left\{-\,\mathrm{i}\int\mathrm{d}z\;V\,[\,\delta\,/\,\mathrm{i}\,\delta\,J_z\,]\right\}\;Z_0\,[\,J=0\,]\\ &\stackrel{\rm def}{=}\;:\exp\left\{\mathrm{i}\int\mathrm{d}x\;\phi_{\rm int}\,(x)\,\mathcal{K}_x\;(\,\delta\,/\,\mathrm{i}\,\delta\,J_x\,)\right\}:\;Z\,[\,J=0\,]\end{aligned}$$

in which I have used in the last step the perturbative definition (1.4) of the generating functional for the Green's functions of the real self-interacting scalar field theory. From the exponential Taylor's expansion (1.1)

$$\begin{aligned} Z\,[\,J\,] &= \left\langle T\,\exp\left\{\mathrm{i}\int\mathrm{d}x\;\phi(x)\,J(x)\right\}\right\rangle_0\\ &\stackrel{\rm def}{=} \sum_{n=0}^{\infty}\frac{\mathrm{i}^{\,n}}{n!}\int\mathrm{d}x_1\;J(x_1)\cdots\int\mathrm{d}x_n\;J(x_n)\\ &\times\;\;\langle\,0\,|\,T\,\phi\,(x_1)\cdots\phi\,(x_n)\,|\,0\,\rangle \end{aligned} \tag{2.12}$$

we eventually come to the fundamental formula that relates the scattering operator, which describes spinless massive particles collisions, to the $n-$point Green's functions of the real self-interacting scalar field theory : namely,

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\frac{\mathrm{i}^{\,n}}{n\,!}\int\mathrm{d}x_1\int\mathrm{d}x_2\;\ldots\;\int\mathrm{d}x_n\\ &:\,\phi_{\rm int}\,(x_1)\,\phi_{\rm int}\,(x_2)\;\ldots\;\phi_{\rm int}\,(x_n)\;:\\ &\mathcal{K}\,(x_1)\,\mathcal{K}\,(x_2)\;\ldots\;\mathcal{K}\,(x_n)\\ &\langle\,0\,|\,T\,\phi\,(x_1)\,\phi\,(x_2)\;\ldots\;\phi\,(x_n)\,|\,0\,\rangle \end{aligned} \tag{2.13}$$

in which $\mathcal{K}\,(x_\jmath\,) = (\Box_\jmath + m^2\,)\quad(\,\jmath = 1,2,\ldots,n\,)$ is the kinetic diffrential operator, *i.e.* specifically the Klein-Gordon operator.

The calculation of the matrix elements of the scattering operator is now straightforward. Let the initial state involve N identical spinless massive particles while the final state N' of such a kind of particles : then we have

$$|\, k_1 k_2 \cdots k_N \,\rangle = (N!)^{-1/2} \prod_{\imath=1}^{N} a^{\dagger}(k_\imath)\,|\,0\,\rangle$$

$$\langle\, k_1' k_2' \cdots k_{N'}' \,| = \langle\, 0\,| \prod_{\jmath=1}^{N'} a(k_\jmath')\,(N'!)^{-1/2}$$

If we suppose that $k_\imath \neq k_\jmath'$ for all pairs[1] $(\imath\,,\,\jmath)$, then solely the term in the series with

$$:\phi_{\text{int}}(x_1')\,\phi_{\text{int}}(x_2')\ \ldots\ \phi_{\text{int}}(x_{N'}')\,\phi_{\text{int}}(x_1)\,\phi_{\text{int}}(x_2)\ \ldots\ \phi_{\text{int}}(x_N):$$

will indeed contribute – see Problem. Among the $(N+N')$ field operators in the interaction representation, N will act with their destruction parts

$$\phi_{\text{int}}^{(-)}(x_\imath)\ (\imath = 1, 2, \cdots, N)$$

whilst N' with their creation parts

$$\phi_{\text{int}}^{(+)}(x_\jmath')\ (\jmath = 1, 2, \cdots, N')$$

This produces a combinatorial factor

$$\binom{N+N'}{N'} = \frac{(N+N')!}{N!\,N'!}$$

Finally, in the present case of identical spinless massive particles, there are $(N!\,N'!)$ ways of matching the destruction and creation parts of the field operators with the initial and final particles. The result is that the only non-vanishing matrix element is provided by

$$\begin{aligned}&\langle\, k_1' \cdots k_{N'}' \,|\,\phi_{\text{int}}^{(+)}(x_1')\ \ldots\ldots\ \phi_{\text{int}}^{(-)}(x_N)\,|\,k_1 \cdots k_N\,\rangle =\\ &\exp\{-\mathrm{i}\,k_1\cdot x_1\ \cdots + \mathrm{i}\,k_{N'}'\cdot x_{N'}'\} + \text{permutations}\end{aligned}$$

all the others being equal to zero. Hence, turning back to the expression (2.13), taking judiciously into accont the symmetry factors we come to the following form of the matrix element : namely,

$$\langle\, k_1' k_2' \cdots k_{N'}' \,|\, S \,|\, k_1 k_2 \cdots k_N \,\rangle =$$

[1]This means that we disregard the case in which any of the incident particles is not scattered at all.

$$
\begin{aligned}
& \mathrm{i}^{N+N'}\,(N!\,N'!)^{-1/2}\int \mathrm{d}x'_1\ldots\int \mathrm{d}x'_{N'}\int \mathrm{d}x_1\ldots\int \mathrm{d}x_N \\
& \exp\{-\mathrm{i}\,k_1\cdot x_1-\cdots-\mathrm{i}\,k_N\cdot x_N+\mathrm{i}\,k'_1\cdot x'_1+\cdots+\mathrm{i}\,k'_{N'}\cdot x'_{N'}\} \\
& \left(\Box_{1'}+m^2\right)\ldots\left(\Box_{N'}+m^2\right)\left(\Box_1+m^2\right)\ldots\left(\Box_N+m^2\right) \\
& G_{N+N'}\,(x_1,\ldots,x_N\,;\,x'_1,\ldots,x'_{N'})
\end{aligned}
\tag{2.14}
$$

in which all the temporal components of the incoming and outgoing momenta are understood to be *on the mass shell*, *i.e.* ,

$$
k_\imath^0=\omega(\mathbf{k}_\imath) \qquad (\imath=1,2,\ldots,N)
$$

$$
k_\jmath'^0=\omega(\mathbf{k}'_\jmath) \qquad (\jmath=1,2,\ldots,N')
$$

It is now convenient to introduce the Green's functions in momentum space according to the standard definition

$$
\begin{aligned}
& \langle\,0\,|\,T\,\phi(x_1)\,\phi(x_2)\ \ldots\ \phi(x_n)\,|\,0\,\rangle=\int\frac{\mathrm{d}^4p_1}{(2\pi)^4}\int\frac{\mathrm{d}^4p_2}{(2\pi)^4}\ \cdots\ \int\frac{\mathrm{d}^4p_n}{(2\pi)^4} \\
& (2\pi)^4\,\delta(P)\,\widetilde{G}_n\,(p_1,p_2\ldots,p_n)\prod_{\jmath=1}^{n}\exp\{-\mathrm{i}\,p_\jmath\cdot x_\jmath\} \\
& =\ G_n\,(x_1,x_2\ldots,x_n)
\end{aligned}
\tag{2.15}
$$

where the $\delta-$distribution of the total energy-momentum

$$
P\ \equiv\ p_1+p_2+\ \ldots\ +p_n
$$

does vindicate the translation invariance of the $n-$point Green's functions in the configuration space. Notice that the canonical engeneering dimensions of the Green's functions in natural units are

$$
[\,G_n\,(x_1\ldots,x_n)\,]=\mathrm{eV}^n \qquad [\,\widetilde{G}_n\,(p_1\ldots,p_n)\,]=\mathrm{cm}^{3n-4}
$$

Then, subtstituting the Fourier transform (2.15), it is straightforward to recast the above equation (2.14) into the final form

$$
\begin{aligned}
& \langle\,k'_1\,k'_2\cdots k'_{N'}\,|\,S\,|\,k_1\,k_2\cdots k_N\,\rangle\ = \\
& i^{N+N'}\,(N!\,N'!)^{-1/2}\ (2\pi)^4\,\delta(K_i-K'_f)\ \times \\
& \prod_{\jmath=1}^{N'}\lim_{k_\jmath'^{\,2}\to m^2}\left(m^2-k_\jmath'^{\,2}\right)\prod_{\imath=1}^{N}\lim_{k_\imath^2\to m^2}\left(m^2-k_\imath^2\right) \\
& \times\,\widetilde{G}_{N+N'}\,(k'_1,\ldots,k'_{N'}\,;\,-k_1,\ldots,-k_N)
\end{aligned}
\tag{2.16}
$$

This remarkable formula is known as the *LSZ reduction formula*, the acronymus being associated to the names of Harry Lehmann, Kurt Symanzik and Wolfhart Zimmermann who firstly obtained that fundamental relation.

If the initial and final states do not truly correspond to sharply definite values of energy and momentum, then some normalized wave packets have to be suitably introduced according to

$$|\,N \text{ initial}\,\rangle \;=\; (\,N\,!\,)^{-1/2}\,\prod_{\imath=1}^{N}\int D\,k_\imath\;\widetilde{f}_\imath(\,k_\imath\,)\,a^\dagger\,(\,k_\imath\,)\;|\,0\,\rangle$$

$$\langle\,N^{\,\prime} \text{ final}\,|\;=\;(\,N^{\,\prime}\,!\,)^{-1/2}\,\prod_{\jmath=1}^{N^{\,\prime}}\int Dk_\jmath^\prime\;\widetilde{f}_\jmath^{\,*}\,(k_\jmath^\prime)\;\langle\,0\,|\;a\,(\,k_\jmath^\prime)$$

where I used the previously introduced notation $[\,k^0=\omega\,(\mathbf{k})=\surd(\mathbf{k}^2+m^2\,)\,]$

$$\begin{aligned} D\,k_\imath \;&=\; \frac{\mathrm{d}^3\mathbf{k}_\imath}{(2\pi)^3\,2\,k_\imath^0} \;=\; \frac{\mathrm{d}^3\mathbf{k}_\imath}{(2\pi)^3\,2\,\omega\,(\mathbf{k}_\imath)} \\ D\,k_\jmath^\prime \;&=\; \frac{\mathrm{d}^3\mathbf{k}_\jmath^\prime}{(2\pi)^3\,2\,k_\jmath^{0\,\prime}} \;=\; \frac{\mathrm{d}^3\mathbf{k}_\jmath^\prime}{(2\pi)^3\,2\,\omega\,(\mathbf{k}_\jmath^\prime)} \end{aligned} \tag{2.17}$$

$$\left[\,a\,(k)\,,\,a^\dagger\,(k^{\,\prime}\,)\,\right]=(2\pi)^3\,2\omega_{\mathbf{k}}\,\delta\,(\,\mathbf{k}-\mathbf{k}^{\,\prime}\,)\qquad\text{et cetera}$$

$$f(t,\mathbf{x})=\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\,2\,\omega\,(\mathbf{k})}\;\widetilde{f}(\mathbf{k})\;\exp\{-\,\mathrm{i}\,t\,\omega\,(\mathbf{k})+\,\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}\}$$

$$(\,f\,,\,f\,)=\int\mathrm{d}^3\mathbf{x}\;f^{\,*}(t,\mathbf{x})\,\mathrm{i}\,\overleftrightarrow{\partial}_0\,f(t,\mathbf{x})=\int Dk\;|\,\widetilde{f}(k)\,|^2=1$$

The reduction formula for the perturbative scattering operator becomes

$$\begin{aligned} &\langle\,N^{\,\prime}\text{ final}\,|\;S\;|\,N\text{ initial}\,\rangle \;=\; \frac{\mathrm{i}^{\,N+N^{\,\prime}}}{\surd\,(\,N\,!\,N^{\prime}\,!\,)}\;\times \\ &\prod_{\jmath=1}^{N^{\,\prime}}\int Dk_\jmath^\prime\;f_\jmath^{\,*}\,(k_\jmath^\prime)\;\lim_{k_\jmath^{\prime\,2}\to m^2}\;\left(m^2-k_\jmath^{\prime\,2}\right) \\ &\prod_{\imath=1}^{N}\int Dk_\imath\;f_\imath\,(k_\imath)\;\lim_{k_\imath^2\to m^2}\;\left(m^2-k_\imath^2\right) \\ &\times\;\widetilde{G}_{\,N+N^{\,\prime}}\,(\,k_1^\prime,\,\ldots\,,k_{N^{\,\prime}}^\prime\,;\,-\,k_1,\,\ldots\,,-\,k_N\,)\;\;(2\pi)^4\,\delta\,(\,K_i-K_f^\prime\,) \end{aligned} \tag{2.18}$$

The disconnected $n-$point Green's functions do involve also trivial parts, that correspond to the absence of any scattering process. Hence, what we are really interested for is the reduction formula for the *connected* Green's functions, that means, the truly interacting part which actually contribute to the scattering amplitudes. For example, in the 4-point Green's function we find terms which are related to the products of two 2-point Green's functions, *i.e.* two full propagators : such terms do not describe neither scattering nor interaction. To see this, I first decompose the 4-point Green's function into disconnected and connected parts as shown graphically in Fig. N. 8 The first three graphs represent the unscattered or *straight through* or even *forward* propagation of the particles, albeit with fully interacting or *dressed* propagators, *i.e.* 2-point Green's functions that include all order radiative corrections which describe emission and absorbtion of virtual particles, in accordance with the energy-time uncertainty relation of quantum mechanics.

The final graph represents the processes that give rise to the scattering, once we have again removed the four dressed propagator factors to define an amplitude which is named *truncated* or *amputated* 4-point Green's function.

In conclusion, from the reduction formulæ we have learned that the basic ingradients we have to build up in perturbation theory by means of the Feynman rules, in the aim of computing the scattering cross sections to be compared with the experimental data, are the *connected, truncated, on-shell* $n-$*point Green's functions in momentum space.*

One of the fundamental inadequacies of the previously discussed and presently known as the customary perturbative approach to the quantum field theory of truly interacting field is the necessity to introduce into the formulation fictitious non-interacting particles, states and fields and to treat the interaction as some additional small perturbation, which slightly modifies the dynamical quantum system and which may be switched-on or switched-off *ad hoc* and *ad lib.* At first glance it might appear that this procedure does not give rise to any basis for criticism of the theory. Indeed we know that the elementary particles interact intensively with each other only if they are extremely close, tipically at a relative distance of few fm. Therefore, it would appear that at large distances among the particles, where large might have the realistic size of few μm, the field interaction could be disregarded and in a certain reliable approximation it is legitimate to neglect it and to regard the particles realistically as being free.

However, by arguing in this way, we omit from consideration the crucial fact that the particles *continuosly interact with the vacuum*, as it were a sort of a *material medium* through which the particles move. This is a typical quantum mechanical and relativistic effect, an unavoidable consequence of the Heisenberg uncertainty relations and of the mass-energy equivalence in the theory of relativity.

It therefore appears to be rather longing for a development of the theory to deal from the outset with real interacting particles and to avoid carefully the introduction of such a kind of artificious concepts like the fictitious free particles, fields and corresponding quantum states. As a matter of fact, the separation between the free and interacting parts of the total hamiltonian, as well as the very existence of a well defined total hamiltonian operator, are non-covariant and frame dependent assumptions. The free hamiltonian H_0 would be an ill-defined part of the energy-momentum vector, the generator of the space-time translations, it won't be neither conserved in time nor referable to any observable quantity. For these reasons the interaction picture in the quantum field theory is merely a poorly defined fictitious device to recover the scattering matrix and the reduction formulas. Actually, it has been rigorously proved by Rudolf Haag [2] that the interaction picture does not exist at all in quantum field theory, once a few very basic and general features are postulated about the nature of the interacting fields, *viz.* covariance, locality, microcausality and spectrum conditions.

The above sketched serious criticisms have led to some important new developments, *i.e.* a non-perturbative formulation of the collision theory for massive interacting fields, that will be summarized in the next section, the main pillars of which are the Lehmann-Symanzik-Zimmermann asymptotic conditions and the adiabatic switching of the interaction.

[2] R. Haag, *On quantum field theories*, Det Kongelige Danske Videnskabernes Selskab Matematisk-Fysiske Meddelelser **29** (1955) nr. 12, 1-37

2.3 Cross section

2.3.1 Scattering amplitude

In this section we shall consider, for the sake of simplicity, a self-interacting real scalar field describing spinless neutral particles without further internal structure. It is useful to express the $S-$matrix in the form

$$S = \mathbb{I} + \mathrm{i}\,T \tag{2.19}$$

where the unit operator $\mathbb{I}$ is related to unscattered, forward, straight through particle propagation, while T is the *transition matrix*, the matrix elements of which do non-trivially depend upon the field interaction. Then the S–matrix elements are defined to be

$$S_{fi} = \langle\, f \,|\, S \,|\, \mathrm{i} \,\rangle = \delta_{fi} + (2\pi)^4\, \mathrm{i}\, \delta\left(P_f^{\,\prime} - P_i\right)\, \mathcal{M}_{fi} \tag{2.20}$$

where the *invariant $T-$matrix elements* $\mathcal{M}_{fi}$ have been introduced for the scattering process $1 + 2 + \ldots + M \ \longmapsto\ 1' + 2\,' + \ldots + N$: namely

$$\langle\, p_1'\, p_2' \,\ldots\, p_N' \,|\, T \,|\, p_1\, p_2 \,\ldots\, p_M \,\rangle = (2\pi)^4\, \delta\left(P_f^{\prime} - P_i\right)\, \mathcal{M}\,(\, p_i \mapsto p_f') \tag{2.21}$$

in which the energy-momentum 1-particle eigenstates of all the particles are normalized according to the standard covariant convention

$$\langle\, p' \,|\, p \,\rangle \;=\; (2\pi)^3\, 2\omega_{\mathbf{p}}\, \delta\,(\,\mathbf{p} - \mathbf{p}'\,) \qquad\qquad p^0 = \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$$

whilst

$$P_i = \sum_{j=1}^{M} p_j\ , \qquad P_f' = \sum_{k=1}^{N} p_k'\ , \tag{2.22}$$

are the total momenta of the M incoming and N outgoing particles. It turns out that the *scattering amplitude* for the process $M_{\mathrm{in}} \ \longmapsto\ N_{\mathrm{out}}$ will be given by the dimensionless complex quantity

$$\begin{aligned}
&\mathfrak{A}\,(f_1, f_2, \ldots, f_M \ \longmapsto\ g_1, g_2, \ldots, g_N) \;=\\
&\prod_{j=1}^{M} \int \frac{\mathrm{d}\,\mathbf{p}_j}{(2\pi)^3\, 2p_j^0}\ \prod_{k=1}^{N} \int \frac{\mathrm{d}\,\mathbf{p}_k'}{(2\pi)^3\, 2p_k^{0\,\prime}}\ (2\pi)^4\, \mathrm{i}\, \delta\left(P_f' - P_i\right)\\
&\times\ f_j(\mathbf{p}_j)\ g_k^*(\mathbf{p}_k')\ \mathcal{M}(p_1, \ldots, p_M\, ;\, p_1', \ldots, p_N')
\end{aligned} \tag{2.23}$$

where $p_j^0 = \omega(\mathbf{p}_j)$, $p_k^{0\,\prime} = \omega(\mathbf{p}_k^\prime)$ are the dispersion laws of the positive energy incoming and outgoing particles, whereas

$$\begin{aligned} f_\jmath(x) &= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}\mathbf{p}_\jmath}{2\,\omega(\mathbf{p}_\jmath)}\; f_\jmath(\mathbf{p}_\jmath)\,\exp\{-\,\mathrm{i}\,p_\jmath\cdot x\} \qquad \jmath = 1,\ldots,M \\ g_k(x) &= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}\mathbf{p}_k}{2\,\omega(\mathbf{p}_k)}\; g_k(\mathbf{p}_k)\,\exp\{-\,\mathrm{i}\,p_k\cdot x\} \qquad k = 1,\ldots,N \end{aligned}$$

are the particle wave functions. Let me now consider the scattering process

$$1 + 2 \;\longmapsto\; 1' + 2' + \ldots + N' \tag{2.24}$$

in which the spinless particles of the final state are supposed to be in a sharply defined eigenstate of the energy-momentum. Then the quantity

$$\begin{aligned} &\mathfrak{A}(f_1, f_2 \longmapsto 1, 2, \ldots, N) \;= \\ &\prod_{j=1}^{2}\int \mathrm{d}\mathbf{p}_j\,[\,(2\pi)^3\,2\,\omega(\mathbf{p}_j)\,]^{-1/2}\,f_j(\mathbf{p}_j)\;\prod_{k=1}^{N}\int D p_k^\prime\;g_k^*(\mathbf{p}_k^\prime) \\ &\times\,(2\pi)^4\;\mathrm{i}\,\delta\left(P_f^\prime - p_1 - p_2\right)\;\mathcal{M}\,(p_1\,,\,p_2\,;\,p_1^\prime,\ldots,p_N^\prime) \end{aligned} \tag{2.25}$$

will represent *the amplitude of the process in which there are N particles in the final state with wave packets $g_1, g_2, \ldots, g_N$ for two incoming particles with wave packets f_1 and f_2*. Here, as usual, the invariant measure $D\,p_k^\prime$ is provided by equation (2.17).

It is convenient to introduce the quantity

$$\begin{aligned} F(P_f) \;&\equiv\; \prod_{j=1}^{2}\int \mathrm{d}\mathbf{p}_j\,[\,(2\pi)^3\,2\,\omega(\mathbf{p}_j)\,]^{-1}\,f_j(\mathbf{p}_j)\,(2\pi)^4\,\delta\,(P_f - p_1 - p_2) \\ &=\; \prod_{j=1}^{2}\int \mathrm{d}\mathbf{p}_j\,[\,(2\pi)^3\,2\,\omega(\mathbf{p}_j)\,]^{-1}\,f_j(\mathbf{p}_j) \\ &\times\; \int \mathrm{d}^4x\;\;\exp\{i\,(P_f - p_1 - p_2)\cdot x\} \\ &=\; \prod_{j=1}^{2}\int \mathrm{d}^4x\;\;\exp\{\,i\,x\cdot P_f\,\} \\ &\times\; \int \mathrm{d}\mathbf{p}_j\,[\,(2\pi)^3\,2\,\omega(\mathbf{p}_j)\,]^{-1}\,f_j(\mathbf{p}_j)\;\exp\{-\,\mathrm{i}\,p_j\cdot x\} \\ &=\; \prod_{j=1}^{2}\int \mathrm{d}^4x\;\,f_j(x)\;\exp\{i\,(p_1^\prime + p_2^\prime + \ldots + p_N^\prime)\cdot x\,\} \end{aligned} \tag{2.26}$$

It follows therefrom that, if the matrix elements of the transition matrix are smooth functions of the incident momenta, for two very narrow wave packets f_1 and f_2 centered around $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ respectively, we can write down the *differential probability of the process in which there are N particles in the final state within the momentum space infinitesimal volume elements* $d\mathbf{p}'_k$ *around* $\mathbf{p}'_k$, which evidently reads

$$\begin{aligned} & \mathrm{d}\,W\,(f_1, f_2 \longmapsto 1, 2, \ldots, N) \;\approx \\ & |\,F(P_f)\,\mathcal{M}\,(\kappa_1, \kappa_2\,;\,p'_1, \ldots, p'_N)\,|^2 \prod_{k=1}^{N} \frac{\mathrm{d}\,\mathbf{p}'_k}{(2\pi)^3\,2\omega\,(\mathbf{p}'_k)} \end{aligned} \tag{2.27}$$

with $\kappa^0_j = \omega(\boldsymbol{\kappa}_j)$. Notice that we also have, for very narrow wave packets centered around $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ respectively,

$$\begin{aligned} & \frac{1}{(2\pi)^4} \int \mathrm{d}^4 P_f\; |\,F(P_f)\,|^2 = \int \mathrm{d}^4 x\; |\,f_1(x)\,f_2\,(x)\,|^2 \\ & \approx [\,4\,\omega(\boldsymbol{\kappa}_1)\,\omega(\boldsymbol{\kappa}_2)\,]^{-1} \int \mathrm{d}^4 x\;\varrho_1(x)\,\varrho_2(x) \end{aligned} \tag{2.28}$$

so that we can approximately set

$$\begin{aligned} |\,F(P_f)\,|^2 \;&\approx\; (2\pi)^4\,\delta\,(P_f - \kappa_1 - \kappa_2) \\ &\times\; [\,4\,\omega(\boldsymbol{\kappa}_1)\,\omega(\boldsymbol{\kappa}_2)\,]^{-1} \int \mathrm{d}x\;\varrho_1(x)\varrho_2(x) \end{aligned}$$

As a consequence we eventually find that, for very narrow incoming wave packets centered around $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ respectively, we can safely write

$$\mathrm{d}\,W\,(f_1, f_2 \longmapsto 1, 2, \ldots, N) \;\approx \tag{2.29}$$

$$\begin{aligned} & (2\pi)^4\,\delta\,(P_f - \kappa_1 - \kappa_2)\;|\,\mathcal{M}\,(\kappa_1, \kappa_2\,;\,p'_1, \ldots, p'_N\,)\,|^2 \\ \times\;\; & \tfrac{1}{4} \prod_{\imath=1}^{2} \int \mathrm{d}^4 x\;[\,\varrho_\imath\,(x)\,/\,\omega\,(\boldsymbol{\kappa}_\imath)\,] \prod_{k=1}^{N} \frac{\mathrm{d}\,\mathbf{p}'_k}{(2\pi)^3\,2\omega\,(\mathbf{p}'_k)} \end{aligned} \tag{2.30}$$

It is important to gather that the quantity

$$\int \mathrm{d}^4 x\;[\,\varrho_1(x)\,\varrho_2(x)/\,4\,\omega(\boldsymbol{\kappa}_1)\,\omega(\boldsymbol{\kappa}_2)\,] \tag{2.31}$$

is *dimensionless*. In order to compare different experiments, *e.g.* in a large high energy collider, it is convenient to define a quantity which does not depend upon the details of the wave functions of the incoming particles: *the differential cross section* : namely,

$$\begin{aligned} \mathrm{d}\sigma &\equiv \frac{\omega(\boldsymbol{\kappa}_1)\,\omega(\boldsymbol{\kappa}_2)\;\mathrm{d}W}{\int \mathrm{d}x\,\varrho_1(x)\,\varrho_2(x)}\;[\,(\kappa_1\cdot\kappa_2)^2 - m_1^2\,m_2^2\,]^{-1/2} \\ &= \tfrac{1}{4}\,[\,(\kappa_1\cdot\kappa_2)^2 - m_1^2\,m_2^2\,]^{-1/2}\;|\,\mathcal{M}\,(\kappa_1,\kappa_2\,;\,p_1',\ldots,p_N'\,)\,|^2 \\ &\times\;(2\pi)^4\,\delta\,(P_f-\kappa_1-\kappa_2)\;\prod_{k=1}^{N}\frac{\mathrm{d}\,\mathbf{p}_k'}{(2\pi)^3\,2\,\omega\,(\mathbf{p}_k')} \end{aligned} \tag{2.32}$$

which has the dimensions of a surface area and turns out to be manifestly Lorentz invariant.

2.3.2 Luminosity

In an actual scattering experiment one has the situation in which two particle beams collide, or one beam scatters off some fixed target. In those cases the densities $\varrho_1(x)$ and $\varrho_2(x)$ equal the particle densities in the beams and/or in the target, up to a normalization constant which accounts of the beams and/or target geometric structures. With such a kind of normalization, the proportionality factor in eq. (2.32) is then the *integrated luminosity*, which turns out to be Lorentz invariant : namely,

$$\mathrm{d}\sigma \;\equiv\; \mathrm{d}W\left(\int_{-\infty}^{\infty}\mathrm{d}t\;\mathfrak{L}\right)^{-1} \tag{2.33}$$

$$\int_{-\infty}^{\infty}\mathrm{d}t\;\mathfrak{L}\;\equiv\;\sqrt{(\kappa_1\cdot\kappa_2)^2-m_1^2m_2^2}\;\int\mathrm{d}^4x\;\frac{\varrho_1(x)}{\omega\,(\boldsymbol{\kappa}_1)}\cdot\frac{\varrho_2(x)}{\omega\,(\boldsymbol{\kappa}_2)} \tag{2.34}$$

for reactions with two incoming massive particles. Notice that, in physical units, $[\,\mathrm{d}\sigma\,] = \mathrm{cm}^2$, $[\,\mathfrak{L}\,] = \mathrm{cm}^{-2}\,\mathrm{s}^{-1}$. The luminosity is the quantity that gives a measure of the scattering event rate R, within experimental settings described by two bunches of incident particles with the particle densities $\varrho_\imath(x)$ $(\imath = 1, 2)$ owing to

$$\frac{\mathrm{d}}{\mathrm{d}t}\,N_{\mathrm{out}} \equiv R = \mathrm{d}\sigma\,\mathfrak{L} \tag{2.35}$$

For instance – see C. Amsler *et al.* (Particle Data Group), Physics Letters **B667**, 1 (2008) `http://pdg.lbl.gov` – the luminosity at LHC for proton–proton collisions during the year 2010 is expected to be $\mathfrak{L} \sim 10^{34}\ \mathrm{cm}^{-2}\,\mathrm{s}^{-1}$. In general, present day high energy colliders reach luminosities within the range

$10^{28} \div 10^{34}$ cm^{-2} s^{-1} . The luminosity is one of the most crucial parameters for a colliding beam storage ring accelerator machine. Cross sections are usually of the order of [3] :

- 1 millibarn = 1 mb = 10^{-27} cm^2 for strong interactions
- 1 nanobarn = 1 nb = 10^{-33} cm^2 for electromagnetic interactions
- 1 femtobarn = 1 fb = 10^{-39} cm^2 for weak interactions

Typical event rates in these processes, assuming $\mathfrak{L} \sim 10^{34}$ cm^{-2} s^{-1} are, therefore, of the order 10^7 s^{-1} , 10 s^{-1} , 10^{-5} s^{-1} , respectively. These numbers clearly illustrate the difficulty to measure weak interaction effects in colliding beam experiments.

In order to understand the physical meaning of the cross section and of the luminosity, let us consider for instance the case of a fixed–target experiment in the target rest frame. We have a target of volume V_{target} placed in a particle beam. The particle densities ϱ_1 and ϱ_2 are supposed to be approximately homogeneous and the number of target particles is $N_{\text{target}} = \varrho_2 V_{\text{target}}$, where ϱ_2 is the particle density of the target. Then we have

$$\int \mathrm{d}^4x\, \varrho_1(x)\, \varrho_2(x) = \int_{-\infty}^{\infty} \mathrm{d}t\, \varrho_1\, \varrho_2\, V_{\text{target}} = \int_{-\infty}^{\infty} \mathrm{d}t\, \varrho_1\, N_{\text{target}} \qquad (2.36)$$

$$(\kappa_1 \cdot \kappa_2)^2 - m_1^2 m_2^2 = (\kappa_1^0\, \boldsymbol{\kappa}_2 - \kappa_2^0\, \boldsymbol{\kappa}_1)^2 + (\boldsymbol{\kappa}_1 \cdot \boldsymbol{\kappa}_2)^2 - \boldsymbol{\kappa}_1^2 \boldsymbol{\kappa}_2^2 \qquad (2.37)$$

The last two terms of the second relation can be dropped in the case of collinear momenta, which is the case in the fixed-target rest frame or in the collider storage ring in the center of mass rest frame, so that we can write

$$[\,(\kappa_1 \cdot \kappa_2)^2 - m_1^2 m_2^2\,]^{1/2} = \kappa_1^0\, \kappa_2^0 \left| \frac{\boldsymbol{\kappa}_1}{\omega_1} - \frac{\boldsymbol{\kappa}_2}{\omega_2} \right| = \omega(\boldsymbol{\kappa}_1)\, \omega(\boldsymbol{\kappa}_2)\, v_{\text{rel}} \qquad (2.38)$$

Combining eq.s (2.34),(2.36),(2.38) we eventually find that the luminosity is given by

$$\mathfrak{L} = \varrho_{\text{beam}}\, v_{\text{rel}}\, N_{\text{target}} = \Phi_{\text{beam}}\, N_{\text{target}} \qquad (2.39)$$

where the beam flux Φ_{beam} is defined as the number of incoming particle passing through a unit area orthogonal the relative velocity vector *per* unit of time. The luminosity in a fixed-target experiment is much higher than for colliding beams in a storage ring machine. Typical flux factors are

[3] The standard unit of measure for the cross sections is 1 barn=10^{-24} cm^2 and a typical paradigmatic quantity is the Thomson cross section $\sigma_T = 8\pi r_e^2/3 = 0.665\,245\,873(13)$ barn

- 10^{10} cm^{-2} s^{-1} for hadron beams
- 10^{8} cm^{-2} s^{-1} for electron beams
- 10^{6} cm^{-2} s^{-1} for neutrino beams

whereas a target contains $10^{26} \div 10^{35}$ protons. This explains why the huge number of protons in a target leads to event rates R much higher than those ones in colliding beam machines. Here, if two bunches containing N_1 and N_2 particles collide with frequency f , so that $v_{\rm rel} = f\,R_{\rm ring}$, $R_{\rm ring}$ being the collider mean radius, then the luminosity is approximately given by

$$\mathfrak{L} \approx f\,\frac{N_1 N_2}{4\pi\sigma_h\sigma_v} \tag{2.40}$$

where σ_h and σ_v actually characterize the gaussian transverse beam profile in the horizontal and vertical directions.

2.3.3 Quasi-elastic scattering

To examine the kinematics further on, let me now consider a *quasi*-elastic scattering $1 + 2 \longmapsto 1' + 2'$ of two incident massive scalar particles with masses m_1, m_2 in two final massive scalar particles with masses m_1', m_2' and suppose that all masses are different. Then, in the *center of momentum rest frame* $\mathbf{p}_1 + \mathbf{p}_2 = 0$, we have that the total energy square is given by

$$s \equiv (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2\omega_1(\mathbf{p}_1)\,\omega_2(\mathbf{p}_2) = (p_1^0 + p_2^0)^2 \ , \tag{2.41}$$

$$(p_1 \cdot p_2)^2 - m_1^2 m_2^2 = \frac{1}{4}\,[\,s - (m_1 - m_2)^2\,]\,[\,s - (m_1 + m_2)^2\,] \tag{2.42}$$

Notice that the last quantity vanishes, as it must, at the *reaction threshold* $s = (m_1 + m_2)^2$. Since the total 4-momentum is conserved in the scattering process we evidently obtain

$$s = (p_1 + p_2)^2 = (p_1' + p_2')^2 \tag{2.43}$$

In addition to the square of the total energy in the center of mass frame, it is convenient to define the *invariant 4-momentum transfer* squared variable

$$t \equiv (p_1' - p_1)^2 = (p_2' - p_2)^2 \tag{2.44}$$

together with the *invariant 4-momentum exchange* squared variable

$$u \equiv (p_2' - p_1)^2 = (p_1' - p_2)^2 \tag{2.45}$$

A little algebra shows that

$$s + t + u = m_1^2 + m_2^2 + m_1'^2 + m_2'^2 \tag{2.46}$$

The kinematic relativistically invariant s, t, u are called the *Mandelstam's variables* [4]. For the special case of two particles in the final state, we can nicely simplify the general expression of eq. (2.32), by partially evaluating the phase-space integrals in the center of momentum frame in which

$$\begin{aligned}
&\mathbf{p}_1 + \mathbf{p}_2 = 0 = \mathbf{p}_1' + \mathbf{p}_2' \\
&\mathbf{p} = \mathbf{p}_1 = -\,\mathbf{p}_2\,, \quad \mathbf{p}\,' = \mathbf{p}_1' = -\,\mathbf{p}_2' \\
&E_1 + E_2 = \sqrt{s} = E_1' + E_2'
\end{aligned}$$

so that

$$\begin{aligned}
&I \equiv \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = |\,\mathbf{p}\,|\sqrt{s} \\
&\prod_{k=1}^{2} \int \frac{\mathrm{d}\mathbf{p}_k'}{(2\pi)^3\, 2\omega(\mathbf{p}_k')}\ (2\pi)^4\, \delta\,(p_1' + p_2' - p_1 - p_2) \\
&= \int \frac{\mathrm{d}p'\, p'^2\, \mathrm{d}\Omega_{\mathrm{CM}}}{16\pi^2\, E_1' E_2'}\ \delta\left(\sqrt{s} - E_1'(p') - E_2'(p')\right)
\end{aligned} \tag{2.47}$$

where $\mathrm{d}\,\Omega_{\mathrm{CM}} = \mathrm{d}\,\phi\,\sin\theta\,\mathrm{d}\,\theta$ is the solid angle element of the momentum $\mathbf{p}'$ in the center of momentum frame of the two outgoing particles. Note that this integral vanishes unless $s > (m_1' + m_2')^2$, *i.e.* the incoming energy in the collision must be actually enough to produce two physical particles at rest with masses m_1' and m_2'.

In order to calculate the value of $p\,'(s)$ for which the argument of the $\delta-$distribution vanishes, we have to obtain the inversion formulas

$$\begin{aligned}
s &= \left(\sqrt{\mathbf{p}\,'^2 + m_1'^2} + \sqrt{\mathbf{p}\,'^2 + m_2'^2}\,\right)^2 \\
&= 2\mathbf{p}\,'^2 + m_1'^2 + m_2'^2 + 2\,E_1'(\mathbf{p}\,')\,E_2'(\mathbf{p}\,')
\end{aligned} \tag{2.48}$$

that is

$$\left(s - 2\,\mathbf{p}\,'^2 - m_1'^2 - m_2'^2\right)^2 = 4\left(\mathbf{p}\,'^2 + m_1'^2\right)\left(\mathbf{p}\,'^2 + m_2'^2\right) \tag{2.49}$$

which finally yields

$$|\,\mathbf{p}'\,|^2 = \frac{1}{4s}\,[\,s^2 + (m_1'^2 - m_2'^2)^2\,] - \frac{1}{2}\,(m_1'^2 + m_2'^2) \tag{2.50}$$

$$|\,\mathbf{p}\,|^2 = \frac{1}{4s}\,[\,s^2 + (m_1^2 - m_2^2)^2\,] - \frac{1}{2}\,(m_1^2 + m_2^2) \tag{2.51}$$

[4] Stanley Mandelstam, *Determination of the Pion-Nucleon Scattering Amplitude from Dispersion Relations and Unitarity. General Theory*, Phys. Rev. **112** (1958) 1344

Now we have

$$\int_{-\infty}^{\infty} \mathrm{d}x\ \varphi(x)\,\delta\,(\,f(x)\,) = \frac{\varphi\,(x_*)}{|\,f'\,(x_*)\,|} \qquad\qquad f\,(x_*) = 0 \tag{2.52}$$

and applying it to the integral (2.47) we get

$$\begin{aligned} \frac{1}{16\,\pi^2} \quad & \cdot \quad \frac{p'^{\,2}(s)}{\surd\{\,[\,p'^{\,2}(s)+m_1'^{\,2}\,]\,[\,p'^{\,2}(s)+m_2'^{\,2}\,]\,\}} \\ \times \quad & \left(\frac{p'\,(s)}{\surd[\,p'^{\,2}(s)+m_1'^{\,2}\,]} + \frac{p'\,(s)}{\surd[\,p'^{\,2}(s)+m_2'^{\,2}\,]}\right)^{-1} \int \mathrm{d}\,\Omega_{\,\mathrm{CM}} \\ = \quad & \frac{|\,\mathbf{p}'\,|}{16\pi^2\sqrt{s}} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\,(-\,\cos\theta) \end{aligned} \tag{2.53}$$

where $|\,\mathbf{p}'\,| = p'\,(s)$ can be expressed as a function of the center of momentum total energy s by the inversion formula (2.50).

Hence we eventually obtain the main formula for the differential cross section for the quasi-elastic collision in the center of momentum frame :

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \frac{1}{64\pi^2\,s}\,|\,\mathcal{M}\,(\,s,t,u\,)\,|^{\,2}\,\frac{|\,\mathbf{p}'\,|}{|\,\mathbf{p}\,|} \tag{2.54}$$

It is convenient to introduce the invariant function

$$\begin{aligned} (2\sqrt{s}\,|\,\mathbf{p}'\,|)^{\,2} \;&=\; s^2 + m_1'^{\,4} + m_2'^{\,4} - 2sm_1'^{\,2} - 2sm_2'^{\,2} - 2m_1'^{\,2}m_2'^{\,2} \\ &\equiv\; [\,F(s,m_1'^{\,2},m_2'^{\,2})\,]^{\,2} = 4sp'^{\,2} \end{aligned} \tag{2.55}$$

in such a way that the final 2-particle invariant phase–space volume element in the center of momentum (CM) frame can be expressed by

$$\frac{1}{32\pi^2\,s}\;F\left(\,s,m_1'^{\,2},m_2'^{\,2}\,\right)\;\mathrm{d}\,\Omega_{\mathrm{CM}} \tag{2.56}$$

so that the basic formula (2.32) takes the form

$$\begin{aligned} \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} \;&=\; \frac{1}{4}\,[\,(\kappa_1\cdot\kappa_2)^2 - m_1^2\,m_2^2\,]^{-1/2}\,|\,\mathcal{M}\,(s,t,u)\,|^{\,2} \\ &\times\; \frac{1}{32\pi^2\,s}\;F\left(\,s,m_1'^{\,2},m_2'^{\,2}\,\right) \\ &=\; \frac{1}{64\pi^2 s}\,|\,\mathcal{M}\,(s,t,u)\,|^{\,2}\,\frac{F\,(\,s,m_1'^{\,2},m_2'^{\,2}\,)}{F(s,m_1^2,m_2^2)} \end{aligned} \tag{2.57}$$

which coincides with eq. (2.54). In the limit of equal masses we eventually come to the very suggestive formula

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \frac{|\,\mathcal{M}\,|^{2}}{64\pi^{2}s} \tag{2.58}$$

It is useful to remark that in the case of equal masses the characteristic function $F(s, m_1^2, m_2^2)$ becomes

$$F(s, m^2) = \sqrt{s(s-4m^2)} \geq 0 \quad \Leftrightarrow \quad s \geq 4m^2 \tag{2.59}$$

which is positive as it does only above the threshold, *i.e.* two equal massive particles at rest. By the way, in the case $m_1 = m_2 = m_e\,,\ m_1' = m_2' = m_\mu$ we have

$$\left[\,F\left(s, m_\mu^2\right)/\,F\left(s, m_e^2\right)\right] \;=\; \sqrt{\frac{s-4m_\mu^2}{s-4m_e^2}} \tag{2.60}$$

When $s \sim 400 \div 2000\ \mathrm{GeV}^2$ and taking into account that $(m_e/m_\mu)^2 \approx 23\times 10^{-6}$ we can safely approximate

$$\left[\,F\left(s, m_\mu^2\right)/\,F\left(s, m_e^2\right)\right] \;\simeq\; \sqrt{1-\left(\frac{2m_\mu}{E_{\mathrm{CM}}}\right)^2} \tag{2.61}$$

Let us rewrite once again the basic formulæ (2.54),(2.57) in a further different form in terms of the transfer momentum Mandelstam's variable

$$\begin{aligned} t = (p_1 - p_1')^2 &= m_1^2 + m_1'^2 - 2\,p_1\cdot p_1' \\ &= m_1^2 + m_1'^2 - 2E_1E_1' + 2pp'\cos\theta_1 \end{aligned} \tag{2.62}$$

where θ_1 is the angle between $\mathbf{p}_1$ and $\mathbf{p}_1'$ so that

$$\begin{aligned} &\mathrm{d}t = 2\,p\,p'\ \mathrm{d}\,(\cos\theta_1) \\ &\int \mathrm{d}\,\Omega_{\mathrm{CM}} = \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\,(-\,\cos\theta_1) = \int_0^{2\pi}\int_{-1}^{1}\frac{\mathrm{d}\phi\,\mathrm{d}t}{2\,p\,p'} \end{aligned} \tag{2.63}$$

As a consequence, if the differential cross section does not depend upon the azimutal angle ϕ_1 we can definitely write

$$\begin{aligned} \left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)_{\mathrm{CM}} &= \frac{|\,\mathcal{M}(s,t,u)\,|^{2}}{64\pi\,s\,|\,\mathbf{p}\,|^{2}} \\ &= \frac{1}{16\pi}\,\left(\frac{|\,\mathcal{M}\,(s,t,u)\,|}{F\,(\,s, m_1^2, m_2^2\,)}\right)^{2} \end{aligned} \tag{2.64}$$

As a final important comment, I remark that the generalization of all the above formulas to the case of scattering of particles with spin is really straightforward. In such cases, in fact, the amplitudes have spinor and/or 4-vector indices, which need thereby to be saturated with the corresponding suitable quantities describing the polarization states.

Specifically, for spin $\frac{1}{2}$ Dirac fermions : $u_r(p)$ for an incoming particle, $\bar{v}_r(p)$ for incoming antiparticles, $\bar{u}_r(p)$ for outgoing particles and $v_r(p)$ for an outgoing antiparticle ; for spin one real vector bosons : $e_\mu(k)$ for incoming vector particles, $e_\nu^*(k)$ for outgoing vector particles, both in massive and massless cases. *Et cetera.* Let me discuss some few enlightening examples.

2.4 Electron-positron into $\mu^+\mu^-$ pairs

The annihilation of an electron−positron pair into a muon-antimuon pair is the simplest of all the quantum electrodynamics processes, but also one of the most important in high-energy physics. It turns out to be fundamental to the understanding of all reactions which occur in e^+e^- colliders. As a matter of fact, it is used indeed to calibrate such a kind of machines. The related process of the electron-positron pair annihilation into a quark-antiquark pair is extraordinarily useful and crucial to unravel elementary particle physics properties. Here below, the lowest order unpolarized cross section will be obtained, up to the accuracy for the electron mass can be disregarded with respect to the muon mass – remember that $(m_e/m_\mu) \approx 0.5\%$.

The Feynman rules give at once the lowest order $O(e^2)$ amplitude, see fig. N 9 : *viz.*,

$$\mathrm{i}\,\mathcal{M} \;=\; \bar{v}_{r'}(p')\,\gamma^\mu\,u_r(p)\,\frac{\mathrm{i}\,e^2\,g_{\mu\nu}}{k^2+\mathrm{i}\varepsilon}\,\bar{u}_s(q)\,\gamma^\nu\,v_{s'}(q') \tag{2.65}$$

which is dimensionless, where $p+p'=k=q+q'$ is the virtual (*i.e.* off-shell) photon energy-momentum such that $k^2>0$. To compute the differential cross section we need an expression for the square modulus of the above amplitude (2.65) : we find

$$\left(\bar{v}\,\gamma^\lambda\,u\right)^* = u^\dagger\,\gamma^{\lambda\dagger}\,\gamma^{0\dagger}\,v = u^\dagger\,\gamma^0\,\gamma^\lambda\,\left(\gamma^0\right)^2\,v = \bar{u}\,\gamma^\lambda\,v$$

that vindicates the great advantage of the adjoint spinor notation. Thus the squared matrix element becomes

$$\begin{aligned}|\,\mathcal{M}\,|^2 \;&=\; \frac{e^4}{(k^2)^2}\,g_{\mu\nu}\,g_{\rho\sigma}\\ &\times\;\Big(\bar{v}_{r'}(p')\,\gamma^\mu\,u_r(p)\,\bar{u}_r(p)\,\gamma^\rho\,v_{r'}(p')\Big)\\ &\times\;\Big(\bar{u}_s(q)\,\gamma^\nu\,v_{s'}(q')\,\bar{v}_{s'}(q')\,\gamma^\sigma\,u_s(q)\Big)\end{aligned} \tag{2.66}$$

In this expression any spin state of the involved four fermion Dirac particles is specified. However, in actual experiments it is very difficult or even not possible to keep polarization under control. For instance, one should prepare the initial state from accurately polarized materials and/or analyze the final state using *e.g.* spin dependent multiple scattering.

In most experiments the electron and positrons beams are unpolarized, in such a manner that the measured cross section is an average over the incoming electron and positron polarizations r and r' respectively. On the other side, muon detectors are usually blind to polarization, so that the measured cross section is a sum over the negatively and positively charged muon spin indices s and s' respectively.

In other words, I will be here mainly interested in the squared matrix element, which greatly simplifies when averaged over the initial electron and positron polarizations and further summed over the final muon spins

$$\tfrac{1}{2}\textstyle\sum_{r=1,2}\ \tfrac{1}{2}\sum_{r'=1,2}\ \sum_{s=1,2}\ \sum_{s'=1,2}\ |\,\mathcal{M}\,(r,r'\ \to\ s,s')\,|^2 \tag{2.67}$$

By making use of the completeness relations

$$\sum_{r=1,2}\left\{\begin{array}{l} u_r(\mathbf{p})\otimes\bar{u}_r(\mathbf{p})=\not{p}+m_e \\ v_r(\mathbf{p})\otimes\bar{v}_r(\mathbf{p})=\not{p}-m_e \end{array}\right. \qquad \left(p_0=\sqrt{\mathbf{p}^2+m_e^2}\right)$$

$$\sum_{s=1,2}\left\{\begin{array}{l} u_s(\mathbf{q})\otimes\bar{u}_s(\mathbf{q})=\not{q}+m_\mu \\ v_s(\mathbf{q})\otimes\bar{v}_s(\mathbf{q})=\not{q}-m_\mu \end{array}\right. \qquad \left(q_0=\sqrt{\mathbf{q}^2+m_\mu^2}\right)$$

we readily arrive to

$$\begin{aligned} \tfrac{1}{4}\sum_{r,r'}\sum_{s,s'}|\,\mathcal{M}\,|^2 \ &=\ \frac{e^4}{(k^2)^2}\,g_{\mu\nu}\,g_{\rho\sigma}\ \times \\ \tfrac{1}{4}\,\mathrm{tr}\ [\,(\not{p}\,'-m_e)\,\gamma^\mu\,(\not{p}+m_e)\,\gamma^\rho\,]&\ \mathrm{tr}\ [\,(\not{q}+m_\mu)\,\gamma^\nu\,(\not{q}\,'-m_\mu)\,\gamma^\sigma\,] \end{aligned} \tag{2.68}$$

The general method of calculating traces consists of successive displacements of identical matrix-four-vector. In particular, the trace of an odd number of gamma matrices does vanish, while we easily find

$$\mathrm{tr}\ (\gamma^\mu\gamma^\nu)\ =\ g^{\mu\nu}\,\mathrm{tr}\,\mathbb{I}\ =\ 4\,g^{\mu\nu} \tag{2.69}$$

$$\mathrm{tr}\ \left(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu\right)\ =\ 4\Big(g^{\kappa\lambda}\,g^{\mu\nu}\ -\ g^{\kappa\mu}\,g^{\lambda\nu}\ +\ g^{\kappa\nu}\,g^{\lambda\mu}\Big) \tag{2.70}$$

Hence the e^+e^- trace is

$$4\,\left[\,p'^{\mu}p^{\rho}+p'^{\rho}p^{\mu}-g^{\mu\rho}\,\left(p\cdot p'+m_e^2\right)\,\right]$$

and similarly the muon pair trace yields

$$4\,\left[\,q'^{\sigma}q^{\nu}+q'^{\nu}q^{\sigma}-g^{\nu\sigma}\,\left(q\cdot q'+m_\mu^2\right)\,\right]$$

After contractions of the Lorentz indices we come to the simple expression

$$\begin{aligned} &\tfrac{1}{4}\sum_{r,r'}\sum_{s,s'}|\,\mathcal{M}\,|^2\ =\ \frac{8\,e^4}{(k^2)^2}\ \times \\ &\Big[\,(p\cdot q)(p'\cdot q')+(p\cdot q')(p'\cdot q)+(p\cdot p')\,m_\mu^2 \\ &+\ (q\cdot q')\,m_e^2+2m_\mu^2 m_e^2\,\Big] \end{aligned} \tag{2.71}$$

Neglecting the electron mass, in the center of momentum frame of the e^+e^- and $\mu^+\mu^-$ pairs we have for $m_e \approx 0$

$$\begin{array}{llll} \text{electron :} & \mathbf{p} & p^0 & = \sqrt{\mathbf{p}^2 + m_e^2} \approx |\mathbf{p}| \\ \text{positron :} & \mathbf{p}' = -\mathbf{p} & p' & \end{array}$$

The differential cross section can also be written in the equivalent form, see *e.g.* : C. Amsler *et al.* (Particle Data Group), Physics Letters **B667**, 1 (2008) [`http://pdg.lbl.gov`] *Cross-section formulae for specific processes (Rev.)*, eq. (39.2) p. 1

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \frac{\alpha^2}{4\,s}\,\beta\,\left[\,1+\cos^2\theta+(1-\beta^2)\sin^2\theta\,\right]$$

where $\beta = v/c = |\,\mathbf{q}\,|\,/\,q_0$ is the muon velocity in the center of mass frame which, in a collider, is the laboratory frame too. In the high-energy limit $(\,q_0 \gg m_\mu\,)$ these formulæ reduces to

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} \sim \frac{\alpha^2}{4\,s}\,(\,1+\cos^2\theta\,) \qquad (\,s \gg 4\,m_\mu^2\,)$$

$$\sigma \sim \frac{4\,\pi}{3}\left(\frac{\alpha}{E_{\mathrm{CM}}}\right)^2\left\{1-\frac{3}{8}\left(\frac{m_\mu}{q_0}\right)^4 - \cdots\right\} \qquad (\,E_{\mathrm{CM}} \gg 4\,m_\mu^2\,)$$

In the high-energy regime , $\sqrt{s} = E_{\mathrm{CM}}$ is the only dimensionful quantity in the process, in such a manner that dimensional analysis requires $\sigma \,\propto\, E_{\mathrm{CM}}^{-2}$ and since we knew that $\sigma \,\propto\, \alpha^2$ the only remaining geometric factor to calculate is $4\pi/3$, the volume of the unit 2-sphere.

The remarkable energy dependence of the $e^+e^- \,\rightarrow\, \mu^+\mu^-$ cross-section formula sets the scale for all e^+e^- annihilation processes through a virtual photon and consequent spin 1/2 point-like fermion pairs production

$$e^+e^- \;\rightarrow\; \gamma^* \;\rightarrow\; f\bar{f}$$

At the center of mass square energy it is given by

$$\begin{aligned}\sigma \;\stackrel{\beta\to 1}{\sim}\; N_c\,Q_f^2\,\frac{4\,\pi}{3}\left(\frac{\alpha}{E_{\mathrm{CM}}}\right)^2 \;&=\; N_c\,Q_f^2\,(\hbar\,c)^2\,\frac{86.8\ \text{nanobarns}}{(\,E_{\mathrm{CM}}\ \text{in GeV}\,)^2}\\ &=\; N_c\,Q_f^2\cdot 1\ \text{unit of R}\end{aligned} \qquad (2.75)$$

where $e\,Q_f$ is the fermion charge while N_c is one for leptons and three for quarks, because each quark in the Standard Model appears in three colors. Experimentally, the easiest quantity to measure turns out to be the total rate for the production of all hadrons, the *strongly interacting particles*. The present understanding of strong interactions is provided by a field theory model named *Quantum Chromodynamics*, the non-Abelian generalization of Quantum Electrodynamics, according to which all hadrons are composed of elementary Dirac fermions called quarks

Melinda Y. Han & Yoichiro Nambu
Three-Triplet Model with Double SU(3) Symmetry
The Physical Review **139**, B1006 - B1010 (1965) [Issue 4B – August 1965]
Harald Fritsch, Murray Gell-Mann & Heinrich Leutwyler
Advantages of the color octet gluon picture
Physics Letters **47B** (1973) 365

Quarks appear in a variety of types, named *flavours*, with its own mass and fractional electric charge. Quarks also carry an additional internal quantum label, named *colour*, taking three possible hues : conventionally, red, green and blue. Eventually, colour is the charge of the strong interaction. In Quantum Chromodynamics, the simplest fundamental process which occurs inside all hadrons is

$$e^+e^- \;\to\; \gamma^* \;\to\; q\,\bar{q}$$

namely, the e^+e^- annihilation processes through a virtual photon with the consequent production of a quark-antiquark pair. Once they are created, the strong interaction among quark-antiquark pairs is such that the latter ones combine to form colourless mesons and baryons.

The astonishing feature predicted by Quantum Chromodynamics is that in the high-energy limit the effects of the strong interaction on the quark production processes can be completely neglected : this amazing property is called *asymptotic freedom*

Hugh David Politzer
Reliable Perturbative Results for Strong Interactions?
The Physical Review Letters **30** (1973) 1346–1349 [Issue 26 – June 1973]
David Jonathan Gross & Frank Anthony Wilczek
Asymptotically Free Gauge Theories
The Physical Review D **8**, 3633 - 3652 (1973) [Issue 10 – November 1973]

It is truly quite remarkable that the non-Abelian gauge field theories, based upon special unitary groups, turn out to be the only consistent local and renormalizable models which exhibit the asymptotic freedom in four space-time dimensions.

Asymptotically we expect

$$\sigma\,(e^+e^- \;\to\; \text{hadrons}) \;\overset{\beta\to1}{\sim}\; 3\cdot\Big(\sum_{\text{flavours}} Q_f^2\Big)\,\mathrm{R} \tag{2.76}$$

where the sum runs over all quarks, the masses of which are smaller than $E_{\mathrm{CM}}/2$. When the value of $E_{\mathrm{CM}}/2$ is close the one of the quark masses, then strong interaction cause large deviations from (2.76), the most striking effect being the appearance of bound states just below $E_{\mathrm{CM}} = 2\,m_q$, endorsed by sharp spikes in the cross-section – for an up-to-date review, see W.-M. Yao et al., J. Phys. G **33**, 1 (2006) and 2007 partial update for the 2008 edition available on the PDG WWW pages (http://pdg.lbl.gov/) *Kinematics, Cross-Section Formulae, and Plots.* Actually, experimental measurements between 2.5 and 45 GeV agree quite well with the naïve prediction (2.76) and, in particular, the factor 3 is a strong evidence for the existence of colour

$$\sigma\,(\,e^+e^- \;\to\; \text{hadrons}\,) \;\stackrel{\beta\,\to\,1}{\sim}\; 3\,\cdot\,\left(\frac{4}{9}+\frac{1}{9}+\frac{1}{9}+\frac{4}{9}+\frac{1}{9}\right)\,\mathrm{R} = \frac{11}{3}\,\mathrm{R}$$

2.5 Electron-muon collision

Let me now consider a different although closely related quantum electromagnetic process, the electron-muon scattering

$$e^-\mu^- \;\to\; e^-\mu^-$$

Again, the Feynman rules give at once the lowest order $O\,(e^2\,)$ amplitude, see fig. N 10 : namely,

$$\mathrm{i}\,\mathcal{M} \;=\; \bar{u}_{\,r'}\,(p_1')\,\gamma^\mu\,u_{\,r}\,(p_1)\,\frac{\mathrm{i}\,e^2}{(p_1-p_1')^{\,2}}\,g_{\mu\nu}\,\bar{u}_{\,s'}\,(p_2')\,\gamma^\nu\,u_{\,s}\,(p_2) \quad (2.77)$$

Taking the square modulus as well as the average over the incoming particle spin and the sum over the final particle spin we find

$$\tfrac14\sum_{r,r'}\sum_{s,s'}|\,\mathcal{M}\,|^2 \;=\; \frac14\left(\frac{e^2}{t}\right)^2\; g_{\mu\nu}\,g_{\rho\sigma}\;\times \quad (2.78)$$
$$\mathrm{tr}\;[\,(\,\not p_1'+m_e\,)\,\gamma^\mu\,(\,\not p_1+m_e\,)\,\gamma^\rho\,]\;\mathrm{tr}\;[\,(\,\not p_2'+m_\mu\,)\,\gamma^\nu\,(\,\not p_2+m_\mu\,)\,\gamma^\sigma\,]$$

in which I have employed the momentum trasfer Mandelstam's variable $t = (p_1'-p_1\,)^2$. It is worthwhile to gather that (2.78) coincides with the previous expression (2.68) under the replacements

$$p\;\to\;p_1 \qquad p'\;\to\;-\,p_1' \qquad q\;\to\;p_2' \qquad q'\;\to\;-\,p_2$$

so that, setting once again $m_e\;\approx\;0$,

$$\tfrac14\sum_{r,r'}\sum_{s,s'}|\,\mathcal{M}\,|^2 \;=\; \frac{8\,e^4}{t^2}\;\times$$
$$\big[\,(\,p_1\cdot p_2\,)\,(\,p_1'\cdot p_2'\,)+(\,p_1\cdot p_2'\,)\,(\,p_1'\cdot p_2\,)-(\,p_1\cdot p_1'\,)\,m_\mu^2\,\big] \quad (2.79)$$

This trick, which allows to build up the amplitude of the process

$$e^-\mu^- \;\to\; e^-\mu^-$$

from the knowledge of the amplitude of the related one

$$e^+e^- \;\to\; \mu^+\mu^-$$

is a first example of use of a general rule named *crossing symmetry*. In general, in fact, the $S-$matrix element for any process involving a particle of energy-momentum p in the initial state is equal to the $S-$matrix element for an otherwise identical process, but for the exchange of the antiparticle with 4-momentum $-p$ in the final state.

Conversely, the kinematics in the center of momentum frame will be rather different. Actually we have

$$\begin{aligned}
&\text{incoming electron}: \quad \mathbf{p}_1 = \mathbf{p} \qquad E_1 \approx |\,\mathbf{p}\,| = p \\
&\text{incoming muon}: \quad \mathbf{p}_2 = -\,\mathbf{p} \qquad E_2 = \sqrt{\mathbf{p}^2 + m_\mu^2} = E \\
&\text{outgoing electron}: \quad \mathbf{p}_1' = \mathbf{p}' \qquad E_1' \approx |\,\mathbf{p}'\,| \\
&\text{outgoing muon}: \quad \mathbf{p}_2' = -\,\mathbf{p}' \qquad E_2' = \sqrt{\mathbf{p}'^2 + m_\mu^2} \\
&E_{\mathrm{CM}} = E_1 + E_2 \;\approx\; p + E \;\approx\; E_1' + E_2' \;\Leftrightarrow\; |\,\mathbf{p}'\,| \approx p
\end{aligned}$$

and thereby

$$\begin{aligned}
&p_1 \cdot p_2 = p_1' \cdot p_2' = p\,(p + E\,) \qquad p_1 \cdot p_2' = p_1' \cdot p_2 = p\,(p\,\cos\theta + E\,) \\
&p_1 \cdot p_1' = p^2\,(1 - \cos\theta\,) \qquad t = (p_1 - p_1')^2 \approx -\,2p^2\,(1 - \cos\theta\,)
\end{aligned}$$

in such a manner that we can write

$$\begin{aligned}
\tfrac{1}{4}\sum_{r,r'}\sum_{s,s'}|\,\mathcal{M}\,|^2 \;&=\; \frac{e^4}{2p^4\,\sin^4(\theta/2)} \;\times \\
&\Big[\,(p + E\,)^2 + (\,E + p\cos\theta\,)^2 - m_\mu^2\,(1 - \cos\theta)\,\Big]
\end{aligned} \tag{2.80}$$

Now we can use the basic formula (2.54) with $|\,\mathbf{p}'\,| \approx |\,\mathbf{p}\,| = p$ which yields

$$\begin{aligned}
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} \;&=\; \frac{1}{64\pi^2\,(p + E\,)^2}\cdot\tfrac{1}{4}\sum_{r,r'}\sum_{s,s'}|\,\mathcal{M}\,|^2 \\
&\approx\; \frac{\alpha^2}{2\,E_{\mathrm{CM}}^2\cdot 4\beta^2\,\sin^4(\theta/2)} \\
&\times\;\left[\,(1+\beta\,)^2 + (1 + \beta\,\cos\theta\,)^2 - 2\left(\frac{m_\mu}{E}\right)^2\sin^2\frac{\theta}{2}\,\right]
\end{aligned}$$

where $\beta = v/c \approx p/E$. In the ultra-relativistic limit $E \approx p$ we find

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} \overset{\beta\to 1}{\sim} \frac{\alpha^2}{2p^2\cdot 4\,\sin^4(\theta/2)}\,\Big(1 + \cos^4(\theta/2)\Big)$$

Consider now the very same process in the incident muon rest frame and retaining the electron mass $m_e \ll m_\mu$ but treating the muon mass as very large. Then, if we disregard the muon recoil, the kinematics reads

$$\left.\begin{array}{lcc} \text{incoming electron :} & \mathbf{p}_1 = \mathbf{p} & E_1 = \sqrt{\mathbf{p}^2 + m_e^2} = E \\ \text{incoming muon :} & \mathbf{p}_2 = 0 & E_2 = m_\mu \\ \text{outgoing muon :} & \mathbf{p}_2' \approx 0 & E_2' \approx m_\mu \\ \text{outgoing electron :} & \mathbf{p}_1' = \mathbf{p}' & E_1' \approx E \ll m_\mu \end{array}\right\} \qquad (2.81)$$

with $|\,\mathbf{p}\,| = p \approx |\,\mathbf{p}'\,|$, whence

$$p_1 \cdot p_2 = E\, m_\mu \approx p_1' \cdot p_2' \approx p_1 \cdot p_2' \approx p_1' \cdot p_2$$
$$p_1 \cdot p_1' = E^2 - p^2 \cos\theta \qquad t = (p_1 - p_1')^2 \approx -\,2p^2\,(1 - \cos\theta\,)$$

in such a manner that now we have

$$\sum_{r,r'} \sum_{s,s'} \tfrac14 \,|\,\mathcal{M}\,|^2 = \frac{8\,e^4}{t^2} \times$$
$$\Big[\,(p_1 \cdot p_2\,)(\,p_1' \cdot p_2'\,) + (\,p_1 \cdot p_2'\,)(\,p_1' \cdot p_2\,) - (\,p_1 \cdot p_1'\,)\,m_\mu^2$$
$$-\,(\,p_2 \cdot p_2'\,)\,m_e^2 + 2\,m_\mu^2 m_e^2\,\Big] \qquad (2.82)$$

$$\sum_{r,r'} \sum_{s,s'} \tfrac14 \,|\,\mathcal{M}\,|^2 = \frac{e^4}{2p^4\,\sin^4(\theta/2)} \times$$
$$\Big[\,2\,E^2 m_\mu^2 - m_\mu^2\,(\,E^2 - p^2\,\cos\theta\,) + m_\mu^2 m_e^2\,\Big] \qquad (2.83)$$

If one of the two incident particles is sufficiently heavy, like the muon in the present example, so that its state does not change after the collision, then its role in the process is equivalent to a fixed target for which recoil can be disregarded. Turning back to the main basic formula (2.32) for the differential cross section, in the present case of a 2-particle final state with one very heavy particle we can use the kinematics (2.81) where θ is now the scattering angle of the light particle in the heavy particle rest frame, so that

$$I \equiv \sqrt{(p_1 \cdot p_2)^2 - m^2 M^2} = |\,\mathbf{p}\,|\,M$$
$$\prod_{k=1}^{2} \int \frac{\mathrm{d}\,\mathbf{p}_k'}{(2\pi)^3\, 2\omega(\,\mathbf{p}_k'\,)}\;(2\pi)^4\,\delta\,(p_1' + p_2' - p_1 - p_2)$$
$$= \int \mathrm{d}\Omega\,(\phi,\theta) \int_0^\infty \frac{\mathrm{d}p\,p^2}{16\pi^2 M E\,(p)}\;\delta\left(E_{p'}' - E_p\right)$$
$$= \int \mathrm{d}\Omega\,(\phi,\theta) \int_0^\infty \frac{\mathrm{d}E\,p\,(\,E\,)}{16\pi^2 M}\;\delta\,(E' - E)$$
$$= \frac{|\,\mathbf{p}'\,|}{16\,\pi^2\,M} \int \mathrm{d}\Omega\,(\phi,\theta) \approx \frac{|\,\mathbf{p}\,|}{16\,\pi^2\,M} \int \mathrm{d}\Omega\,(\phi,\theta) \qquad (2.84)$$

Thus, according to the main formula (2.32) and the above fixed target (FT) kinematics, as well as the related final 2-particle phase space integration, we eventually come to the remarkably simple expression

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{FT}} = \frac{|\,\mathcal{M}\,(s,t,u)\,|^{\,2}}{64\,\pi^{2}\,M^{\,2}} \tag{2.85}$$

Inserting the spin averaged and summed amplitude (2.83) and setting $M = m_\mu$ yields

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} = \frac{\alpha^2}{4\,|\,\mathbf{p}\,|^{2}\,\beta^{2}\,\sin^{4}(\theta/2)}\left(1 - \beta^{2}\,\sin^{2}\frac{\theta}{2}\right) \tag{2.86}$$

where $\beta \equiv |\,\mathbf{p}\,|\,/\,E$, which is the celebrated *Mott formula* for the Coulomb scattering of relativistic electrons. In the non-relativistic limit and for a fixed target of atomic number Z we readily recover the *Rutherford formula*. Actually, for $\beta = v\,/\,c\,,\quad \mathbf{p} \approx m\,\mathbf{v}\,,\quad E \approx m\,c^2$, we get the leading term

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Rutherford}} \overset{\beta\to 0}{\sim} \frac{Z^{2}\,\alpha^{2}\,(\hbar c)^{2}}{4\,m^{2}\,v^{4}\,\sin^{4}(\theta/2)} \tag{2.87}$$

2.6 Annihilation of an electron-positron pair

The two-photon annihilation is described, to the lowest order, by the two tree-level diagrams which differ by the two–photon exchange, see fig. N 11. In the center of momentum frame of the e^+e^- pair we have

$$\begin{array}{l} \text{electron}: \quad \mathbf{p}_1 = \mathbf{p} \qquad E_1 = \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2} \\ \text{positron}: \quad \mathbf{p}_2 = -\,\mathbf{p} \qquad E_2 = E_1 = \omega_{\mathbf{p}} \\ \text{first photon}: \quad \mathbf{k}_1 = \mathbf{k}\,, \quad k_1^0 = |\,\mathbf{k}\,| \\ \text{second photon}: \quad \mathbf{k}_2 = -\,\mathbf{k}\,, \quad k_2^0 = k_1^0 = |\,\mathbf{k}\,| \;\equiv\; k_0 \end{array}$$

Moreover, since we will finally sum over both photon polarizations, we associate to the two final photon the real and physical linear polarization vectors

$$\varepsilon_\mu^{\,A}(\mathbf{k}_1) \quad (A = 1,2) \qquad\qquad \varepsilon_\mu^{\,B}(\mathbf{k}_2) \quad (B = 1,2)$$

that satisfy

$$-\,g^{\mu\nu}\,\varepsilon_\mu^{\,A}(\mathbf{k})\,\varepsilon_\nu^{\,B}(\pm\mathbf{k}) = \delta^{AB} \qquad\qquad \jmath\,=\,1,2$$

The Mandelstam's variables are

$$\begin{array}{l} s = (p_1 + p_2)^2 = (k_1 + k_2)^2 = 2m^2 + 2p_1 \cdot p_2 = 2k_1 \cdot k_2 \\ t = (p_1 - k_1)^2 = (p_2 - k_2)^2 = m^2 - 2p_1 \cdot k_1 = m^2 - 2p_2 \cdot k_2 \\ u = (p_1 - k_2)^2 = (p_2 - k_1)^2 = m^2 - 2p_1 \cdot k_2 = m^2 - 2p_2 \cdot k_1 \end{array}$$

with $s+t+u = 2m^2$. Making use of the rules of correspondence we construct the matrix elements

$$\begin{array}{rcl} i\,\mathcal{M} & = & -\,\mathrm{i}e^2\,\varepsilon_\nu\,\varepsilon_\mu\,\bar{v}_r(-\mathbf{p})\,M^{\,\mu\nu}\,u_s(\mathbf{p}) \\ & = & \frac{1}{2}\,\bar{v}_r(\mathbf{p}_2)\,(-ie\,\gamma^\nu)\,\varepsilon_\nu^{\,B}(\mathbf{k}_2)\,S(p_1 - k_1)\,(-ie\,\gamma^\mu)\,\varepsilon_\mu^{\,A}(\mathbf{k}_1)\,u_s(\mathbf{p}_1) \\ & + & [\,\text{photon exchange} \quad \mathbf{k}_1 \;\leftrightarrow\; \mathbf{k}_2 \qquad i \;\leftrightarrow\; j\,, \quad \mu \;\leftrightarrow\; \nu\,] \\ & = & -\,\mathrm{i}e^2\,\bar{v}_r(-\,\mathbf{p})\,Q\,u_s(\mathbf{p})\,[\,(t - m^2)\,(u - m^2)\,]^{-1} \end{array}$$

where we have set

$$Q = (p_1 \cdot k_2)\,\not\varepsilon_2\,(\not p_1 - \not k_1 + m)\,\not\varepsilon_1 + (p_1 \cdot k_1)\,\not\varepsilon_1\,(\not p_1 - \not k_2 + m)\,\not\varepsilon_2$$

Notice that we have

$$\begin{array}{rcl} Q\,u_s(\mathbf{p}) & = & (p_1 \cdot k_2)\,\not\varepsilon_2\,(\not p_1 - \not k_1 + m)\,\not\varepsilon_1\,u_s(\mathbf{p}_1) \\ & + & (p_1 \cdot k_1)\,\not\varepsilon_1\,(\not p_1 - \not k_2 + m)\,\not\varepsilon_2\,u_s(\mathbf{p}_1) \\ & = & (p_1 \cdot k_2)\,\varepsilon_2\,(2p_1 \cdot \varepsilon_1 - \not k_1 \not\varepsilon_1)\,u_s(\mathbf{p}_1) \\ & + & (p_1 \cdot k_1)\,\varepsilon_1\,(2p_1 \cdot \varepsilon_2 - \not k_2 \not\varepsilon_2)\,u_s(\mathbf{p}_1) \end{array}$$

in such a way that we can write

$$\begin{aligned} Q &\doteq (p_1 \cdot k_2)\, \not{\varepsilon}_2 \,[\, 2\,(p_1 \cdot \varepsilon_1) \,-\, \not{k}_1 \not{\varepsilon}_1 \,] \\ &+ (p_1 \cdot k_1)\, \not{\varepsilon}_1 \,[\, 2\,(p_1 \cdot \varepsilon_2) \,-\, \not{k}_2 \not{\varepsilon}_2 \,] \end{aligned}$$

where $\doteq$ means equality up to evanescent terms when acting upon $u\,(\mathbf{p})$. If we are interested in the annihilation of an unpolarized e^+e^- pair then we get

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \frac{\alpha^2}{16\,k_0\,|\,\mathbf{p}\,|} \sum_{r\,,\,s} \frac{|\,\bar{v}_r\,(-\mathbf{p})\,Q\,u_s\,(\mathbf{p})\,|^2}{[\,(\,t - m^2\,)(\,u - m^2\,)\,]^2}$$

To the aim of computing $[\,\bar{v}_r\,(-\mathbf{p})\,Q\,u_s\,(\mathbf{p})\,]^*$ we notice that

$$[\,\bar{v}_r\,(-\mathbf{p})\,Q\,u_s\,(\mathbf{p})\,]^* = \bar{u}_s\,(\mathbf{p})\;\bar{Q}\;\,v_r\,(-\mathbf{p}) \tag{2.88}$$

so that

$$\begin{aligned} \bar{Q} &\doteq (p_1 \cdot k_2)\,[\,2p_1 \cdot e_1 \,-\, \not{\varepsilon}_1 \not{k}_1\,]\,\not{\varepsilon}_2 \\ &+ (p_1 \cdot k_1)\,[\,2p_1 \cdot e_2 \,-\, \not{\varepsilon}_2 \not{k}_2\,]\,\not{\varepsilon}_1 \end{aligned}$$

As a consequence, after summation over spinor polarizations and making use of the property of the ciclicity of the trace operation, we can definitely write

$$\begin{aligned} \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} &= \alpha^2\, \frac{\mathrm{tr}\,[\,(\not{p}_2 - m)\,Q\,(\not{p}_1 + m)\;\bar{Q}\,]}{16\,k_0\,|\,\mathbf{p}\,|\,(\,t - m^2\,)^2\,(\,u - m^2\,)^2} \\ &\equiv \frac{\alpha^2\,\mathrm{tr}\;A}{16\,k_0\,|\,\mathbf{p}\,|}\,[\,(t - m^2)\,(u - m^2)\,]^{-2} \end{aligned} \tag{2.89}$$

where $\mathrm{tr}\,[\,\cdots]$ refers to trace over spinor indices. Making use of the law of the conservation of four-momentum

$$p_2 = k_1 + k_2 - p_1$$

we shall write A in the form of the sum

$$A = A_1 + A_2 + A_3$$

where

$$\begin{aligned} A_1 &= (\not{k}_1 + \not{k}_2)\,Q\,\not{p}_1\;\bar{Q} \\ A_2 &= -\,(\not{p}_1 + m)\,Q\,(\not{p}_1 + m)\;\bar{Q} \\ A_3 &= m\,(\not{k}_1 + \not{k}_2)\,Q\;\bar{Q} \end{aligned}$$

It is immediate to check by direct inspection that A_3 is a sum of products of an odd number of Dirac matrices, whence tr $A_3 = 0$. Moreover, after setting $k_1 + k_2 = k$, we have

$$
\begin{aligned}
\mathrm{tr}\ A_1 &= \mathrm{tr}\ [\,(\not k_1 + \not k_2)\, Q \not p_1\ \bar{Q}\,] \\
&= \mathrm{tr} \Big\{ (p_1 \cdot k_2)\, \not k\, \not\varepsilon_2 [\, 2\,(p_1 \cdot \varepsilon_1) - \not k_1 \not\varepsilon_1\,] \\
&+ \ (p_1 \cdot k_1)\, \not k\, \not\varepsilon_1 [\, 2\,(p_1 \cdot \varepsilon_2) - \not k_2 \not\varepsilon_2 \Big\} \\
&\times \ \Big\{ (p_1 \cdot k_2)\, \not p\, [\, 2\,(p_1 \cdot \varepsilon_1) - \not\varepsilon_1 \not k_1\,] \not\varepsilon_2 \\
&+ \ (p_1 \cdot k_1)\, \not p\, [\, 2\,(p_1 \cdot \varepsilon_2) - \not\varepsilon_2 \not k_2\,] \not\varepsilon_1 \Big\} \\
&= \mathrm{tr} \Big\{ (p_1 \cdot k_2)\, \not k\, \not\varepsilon_2 [\, 2\,(p_1 \cdot \varepsilon_1) - \not k_1 \not\varepsilon_1\,] \Big\} \\
&\times \ \Big\{ (p_1 \cdot k_2)\, \not p\, [\, 2\,(p_1 \cdot \varepsilon_1) - \not\varepsilon_1 \not k_1\,] \not\varepsilon_2 \Big\} \\
&+ \ \mathrm{tr} \Big\{ (p_1 \cdot k_2)\, \not k\, \not\varepsilon_2 [\, 2\,(p_1 \cdot \varepsilon_1) - \not k_1 \not\varepsilon_1\,] \Big\} \\
&\times \ \Big\{ (p_1 \cdot k_1)\, \not p\, [\, 2\,(p_1 \cdot \varepsilon_2) - \not\varepsilon_2 \not k_2\,] \not\varepsilon_1 \Big\} \\
&+ \qquad \Big\{ 1 \ \leftrightarrow\ 2 \Big\}
\end{aligned}
$$

This expression contains the sum of two groups of sixteen terms which are related by the exchange operation $\{1 \ \leftrightarrow\ 2\}$. In turn, each group is done of traces of products of four, six and eight Dirac matrices. The general method of calculating traces consists of successive displacements of identical matrix-four-vector. The calculation of tr A_1 and tr A_2 is then straightforward and elementary, although tedious. For this calculation we shall use equations (5.7) and (5.8) as well as the forthcoming relations, which are a direct consequence of the corresponding definitions: namely,

$$
\begin{aligned}
& k_1^2 = k_2^2 = 0\ ,\ \ k_1 \cdot k_2 = 2(k_1^0)^2 = 2(k_2^0)^2 = 2\,|\,\mathbf{k}\,|^2 \\
& k_1 \cdot \varepsilon_1 = k_2 \cdot \varepsilon_2 = k_1 \cdot \varepsilon_2 = k_2 \cdot \varepsilon_1 = 0 \\
& k_1 \cdot p_1 = k_0^2 - \mathbf{k}_1 \cdot \mathbf{p}_1 = |\,\mathbf{k}\,|\,(\,|\,\mathbf{k}\,| - |\,\mathbf{p}\,|\cos\theta\,) \\
& k_2 \cdot p_1 = k_0^2 - \mathbf{k}_2 \cdot \mathbf{p}_1 = |\,\mathbf{k}\,|\,(\,|\,\mathbf{k}\,| + |\,\mathbf{p}\,|\cos\theta\,)
\end{aligned}
$$

where θ is the angle between the vectors $\mathbf{k}_1$ and $\mathbf{p}_1$. The result is

$$
\begin{aligned}
\mathrm{tr}\ A_1 &= 32\, k_0^2\, (p_1 \cdot k_1)\, (p_1 \cdot k_2) \\
&\times \ [\, 2(\varepsilon_1 \cdot \varepsilon_2)(p_1 \cdot \varepsilon_1)(p_1 \cdot \varepsilon_2) + 4k_0^2 + (p_1 \cdot \varepsilon_1)^2 + (p_1 \cdot \varepsilon_2)^2\,] \\
\mathrm{tr}\ A_2 &= -\, 32\, k_0^2\, (p_1 \cdot k_1)\, (p_1 \cdot k_2)
\end{aligned}
$$

$$
\begin{aligned}
& \times \;\; [\,2(\varepsilon_1\cdot\varepsilon_2)(p_1\cdot\varepsilon_1)(p_1\cdot\varepsilon_2) + (p_1\cdot\varepsilon_1)^2 + (p_1\cdot\varepsilon_2)^2\,] \\
& - \;\; 32\,[\,2k_0^2\,(p_1\cdot\varepsilon_1)(p_1\cdot\varepsilon_2) + (\varepsilon_1\cdot\varepsilon_2)(p_1\cdot k_1)(p_1\cdot k_2)\,]^2 \\
\mathrm{tr}\;A \;&=\;\; 32\,(p_1\cdot k_1)^2\,(p_1\cdot k_2)^2\,\left\{\frac{4k_0^4}{(p_1\cdot k_1)(p_1\cdot k_2)} \;-\; (\varepsilon_1\cdot\varepsilon_2)^2 \;-\right. \\
& -\;\; \left.\frac{4k_0^4\,(p_1\cdot\varepsilon_1)^2(p_1\cdot\varepsilon_2)^2}{(p_1\cdot k_1)^2(p_1\cdot k_2)^2} - \frac{4k_0^2\,(\varepsilon_1\cdot\varepsilon_2)(p_1\cdot\varepsilon_1)(p_1\cdot\varepsilon_2)}{(p_1\cdot k_1)(p_1\cdot k_2)}\right\}
\end{aligned}
$$

By summing over the polarization directions of the photons and by taking into account the relationships

$$
\begin{aligned}
& \sum_{A=1,2}(\,p_1\cdot\varepsilon_1^A\,)^2 = \sum_{A=1,2}(\,p_1\cdot\varepsilon_2^A\,)^2 = |\,\mathbf{p}\,|^{\,2}\sin^2\theta \\
& \sum_{A=1,2}\;\sum_{B=1,2}(\,\varepsilon_1^A\cdot\varepsilon_1^B\,)^2 = 2 \\
& \sum_{A=1,2}\;\sum_{B=1,2}(\,\varepsilon_1^A\cdot\varepsilon_1^B\,)\,(\,p_1\cdot\varepsilon_1^A\,)\,(\,p_1\cdot\varepsilon_2^B\,) = -\,|\,\mathbf{p}\,|^2\sin^2\theta
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\mathrm{tr}\;A \;&=\;\; 4\,(2p_1\cdot k_1)^2\,(2p_1\cdot k_2)^2\;\;\times \\
& \times\;\; \left\{\frac{|\,\mathbf{k}\,|^2 + |\,\mathbf{p}\,|^2\,(1+\sin^2\theta)}{|\,\mathbf{k}\,|^2 - |\,\mathbf{p}\,|^2\cos^2\theta} \;-\; \frac{2|\,\mathbf{p}\,|^4\sin^4\theta}{(\,|\,\mathbf{k}\,|^2 - |\,\mathbf{p}\,|^2\cos^2\theta\,)^2}\right\}
\end{aligned}
$$

In the center of momentum frame we have

$$
2p_1\cdot k_1 = m^2 - t \qquad 2p_1\cdot k_2 = m^2 - u \qquad s = 4|\,\mathbf{k}\,|^2
$$

so that, by substituting the above expression for tr A into eq. (2.89) we obtain the well–known mass–independent formula for the differential cross section for the annihilation of a particle-antiparticle pair first obtained by Paul Adrian Maurice Dirac (1930) : namely,

$$
\begin{aligned}
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} \;&=\;\; \frac{\alpha^2}{4|\,\mathbf{k}\,|\,|\,\mathbf{p}\,|}\left\{\frac{|\,\mathbf{k}\,|^2 + |\,\mathbf{p}\,|^2\,(\,1+\sin^2\theta\,)}{|\,\mathbf{k}\,|^2 - |\,\mathbf{p}\,|^2\cos^2\theta}\right. \\
& -\;\; \left.\frac{2|\,\mathbf{p}\,|^4\sin^4\theta}{(\,|\,\mathbf{k}\,|^2 - |\,\mathbf{p}\,|^2\cos^2\theta\,)^2}\right\}
\end{aligned}
$$

according to [8] § **23** eq. (23.12) p. 282 and [9] § **88** eq. (88,13) p. 431.

2.7 Appendix A

Here we shall first report the so called *heuristic memento derivation* of the differential cross section formula – V.B. Berestetskij, E.M. Lifšits and L.P. Pitaevskij, *Teoria quantistica relativistica*, Editori Riuniti, Roma, 1978, § **65** eq. (65,18) p. 302. From the basic formula

$$S_{fi} = \langle\, f \,|\, S \,|\, \mathrm{i}\,\rangle = \delta_{fi} + (2\pi)^4\, \mathrm{i}\, \delta\,(P'_f - P_i)\, \mathcal{M}(\, p_i \mapsto p'_f)$$

for the scattering process $1 + 2 + \ldots + M \;\longmapsto\; 1' + 2' + \ldots + N'$, we immediately obtain that the transition probability over all space–time is then formally given by

$$dw_{fi} = [\,(2\pi)^4\, \delta\,(P'_f - P_i)\,]^2\, |\, \mathcal{M}(\, p_i \mapsto p'_f)\,|^2\; \mathrm{d}\mathbf{P}_i\, \mathrm{d}\mathbf{P}'_f \tag{2.90}$$

where

$$\mathrm{d}\mathbf{P}_i \;=\; \prod_{j=1}^{M} [\,(2\pi)^3\, 2\omega(\mathbf{p}_j)\,]^{-1}\, \mathrm{d}\mathbf{p}_j \quad \mathrm{d}\mathbf{P}'_f \;=\; \prod_{k=1}^{N} [\,(2\pi)^3\, 2\omega(\mathbf{p}'_k)\,]^{-1}\, \mathrm{d}\mathbf{p}'_k$$

The square of the δ–distribution is understood in the sense that $(2\pi)^4\, \delta^{(4)}(0)$ is nothing but the space–time total volume. This can be readily seen in terms of the formal identities

$$\begin{aligned} 2\pi\, \delta(p) &= \lim_{L\to\infty} \int_{-L}^{L} \mathrm{e}^{\,\mathrm{i}px}\, \mathrm{d}x \\ &= \lim_{L\to\infty} \frac{2}{p}\, \sin(pL) \\ \lim_{p\to 0} 2\pi\delta(p) &= \lim_{L\to\infty} 2L \end{aligned}$$

Notice that in natural units $\hbar = c = 1$ we have the following canonical engeneering dimensions:

$$[\,|\,\mathbf{k}\,\rangle\,] = \mathrm{cm}\,, \qquad [\,\mathcal{M}(\, p_i \mapsto p'_f)\,] = \mathrm{cm}^{\,M+N-4}$$
$$[\,|\,\mathcal{M}(\, p_i \mapsto p'_f)\,|^2\, \delta\,(P'_f - P_i)\,] = \mathrm{cm}^{2M+2N-4}$$

It follows that the transition probability per element of space–time is

$$\mathrm{d}W_{fi} = (2\pi)^4\, \delta\,(P'_f - P_i)\, |\, \mathcal{M}(\, p_i \mapsto p'_f)\,|^2\; \mathrm{d}\mathbf{P}_i\, \mathrm{d}\mathbf{P}'_f \tag{2.91}$$

In scattering experiments one is usually interested in the differential cross section of two incident particles into many. For two incident particles we have

$$
\begin{aligned}
\mathrm{d}W_{fi} &= \frac{1}{16\pi^2}\,\delta\,(P_f' - p_1 - p_2)\,|\,\mathcal{M}(\,p_1, p_2 \mapsto p_f')\,|^{\,2}\,\frac{\Delta\mathbf{p}_1\Delta\mathbf{p}_2}{\omega(\mathbf{p}_1)\omega(\mathbf{p}_2)}\;\mathrm{d}\mathbf{P}_f' \\
&= \tfrac{1}{4}\,(2\pi)^4\,\delta\,(P_f' - p_1 - p_2)\,\frac{|\,\mathcal{M}(\,p_1, p_2 \mapsto p_f')\,|^{\,2}}{\omega(\mathbf{p}_1)\,\omega(\mathbf{p}_2)\,\Delta V_1\,\Delta V_2}\;\mathrm{d}\mathbf{P}_f'
\end{aligned}
$$

where $\Delta\mathbf{p}_1$, $\Delta\mathbf{p}_2$ are very small regions in momentum space centered around $\mathbf{p}_1$ and $\mathbf{p}_2$ respectively. Then, taking eq. (2.38) into account, we eventually find the differential cross section in the form

$$
\begin{aligned}
&\mathrm{d}\sigma\,(1 + 2 \;\longmapsto\; 1' + 2\,' + \ldots + N\,') \\
&\equiv\; \frac{(2\pi)^3}{\Delta\mathbf{p}_1}\cdot\frac{(2\pi)^3}{\Delta\mathbf{p}_2}\cdot\frac{\mathrm{d}W_{fi}}{v_{\mathrm{rel}}} = \frac{1}{\beta_{\mathrm{rel}}}\,\Delta V_1\,\Delta V_2\,\mathrm{d}W_{fi} \\
&=\; \tfrac{1}{4}\,[\,(p_1\cdot p_2)^2 - m_1^2 m_2^2\,]^{-1/2}\;(2\pi)^4\,\delta\,(P_f' - P_i) \\
&\times\;|\,\mathcal{M}(\,p_1, p_2 \mapsto p_f')\,|^{\,2}\prod_{k=1}^{N}[\,(2\pi)^3\,2\omega(\mathbf{p}_k')\,]^{-1}\,\mathrm{d}\mathbf{p}_k' \qquad (2.92)
\end{aligned}
$$

in perfect agreement with eq. (2.32).

2.8 Problems

Evaluate the quantity

$$\langle\, a(k_1')a(k_2') : \phi(x_2')\phi(x_1')\phi(x_1)\phi(x_2) : a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0$$

where $\phi(x) = \phi^{(-)}(x) + \phi^{(+)}(x)$ is a real scalar free field with

$$\phi^{(-)}(x) = \int Dk\, a(k)\, \mathrm{e}^{-\mathrm{i}kx} \qquad \phi^{(+)}(x) = \int Dk\, a^\dagger(k)\, \mathrm{e}^{\mathrm{i}kx}$$

$$k_0 = \omega_{\mathbf{k}} \qquad Dk = \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3\, 2\omega_{\mathbf{k}}} = \frac{\mathrm{d}^4k}{(2\pi)^3}\, \theta(k_0)\, \delta(k^2 - m^2)$$

$$[\, a(k)\, ,\, a^\dagger(p)\,] = (2\pi)^3\, 2\omega_{\mathbf{k}}\, \delta(\mathbf{k} - \mathbf{p}) \qquad [\, a(k)\, ,\, a(p)\,] = 0$$

Solution

Let us first calculate the simpler quantity

$$\begin{aligned}
&\langle\, a(k_1') : \phi(x_1')\phi(x_1) : a^\dagger(k_1)\,\rangle_0 \;= \\
&\langle\, a(k_1')\phi^{(-)}(x_1')\phi^{(-)}(x_1)a^\dagger(k_1)\,\rangle_0 + \langle\, a(k_1')\phi^{(+)}(x_1')\phi^{(+)}(x_1)a^\dagger(k_1)\,\rangle_0 \\
&+ \langle\, a(k_1')\phi^{(+)}(x_1')\phi^{(-)}(x_1)a^\dagger(k_1)\,\rangle_0 + \langle\, a(k_1')\phi^{(+)}(x_1)\phi^{(-)}(x_1')a^\dagger(k_1)\,\rangle_0
\end{aligned}$$

Now we have the commutation relations

$$[\,\phi^{(-)}(x), a^\dagger(k)\,] = e^{-\mathrm{i}kx} \qquad [\, a(k), \phi^{(+)}(x)\,] = e^{\mathrm{i}kx}$$

whence we readily obtain that the first line in the right hand side of the previous equality does vanish, while the very last line yields

$$\begin{aligned}
\langle\, a(k_1') : \phi(x_1')\phi(x_1) : a^\dagger(k_1)\,\rangle_0 \;&=\; \exp\{-\mathrm{i}\, k_1\cdot x_1 + \mathrm{i}\, k_1'\cdot x_1'\} \\
&+\; \exp\{-\mathrm{i}\, k_1\cdot x_1' + \mathrm{i}\, k_1'\cdot x_1\}
\end{aligned}$$

Next we find

$$\begin{aligned}
&\langle\, a(k_1') : \phi(x_1')\phi(x_1)\phi(x_2) : a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;= \\
&\langle\, a(k_1')\phi^{(-)}(x_1')\phi^{(-)}(x_1)\phi^{(-)}(x_2)a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;+ \\
&\langle\, a(k_1')\phi^{(+)}(x_1')\phi^{(-)}(x_1)\phi^{(-)}(x_2)a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;+ \\
&\langle\, a(k_1')\phi^{(+)}(x_1)\phi^{(-)}(x_2)\phi^{(-)}(x_1')a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;+ \\
&\langle\, a(k_1')\phi^{(+)}(x_2)\phi^{(-)}(x_1')\phi^{(-)}(x_1)a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;+ \\
&\langle\, a(k_1')\phi^{(+)}(x_1')\phi^{(+)}(x_1)\phi^{(-)}(x_2)a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;+ \\
&\langle\, a(k_1')\phi^{(+)}(x_1)\phi^{(+)}(x_2)\phi^{(-)}(x_1')a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;+ \\
&\langle\, a(k_1')\phi^{(+)}(x_2)\phi^{(+)}(x_1')\phi^{(-)}(x_1)a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;+ \\
&\langle\, a(k_1')\phi^{(+)}(x_1')\phi^{(+)}(x_1)\phi^{(+)}(x_2)a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0
\end{aligned}$$

The last four lines evidently vanish so that we are left with

$$
\begin{aligned}
&\langle\, a(k_1')\,:\phi(x_1')\phi(x_1)\phi(x_2)\,: a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;= \\
&\langle\, a(k_1')\phi^{(-)}(x_1')\phi^{(-)}(x_1)a^\dagger(k_2)\,\rangle_0 \;\exp\{-\mathrm{i}\,k_1\cdot x_2\} \;+ \\
&\langle\, a(k_1')\phi^{(-)}(x_1')\phi^{(-)}(x_1)a^\dagger(k_1)\,\rangle_0 \;\exp\{-\mathrm{i}\,k_2\cdot x_2\} \;+ \\
&\exp\{\mathrm{i}\,k_1'\cdot x_1'\}\;\langle\, \phi^{(-)}(x_1)\phi^{(-)}(x_2)a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;+ \\
&\exp\{\mathrm{i}\,k_1'\cdot x_1\}\;\langle\, \phi^{(-)}(x_2)\phi^{(-)}(x_1')a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \;+ \\
&\exp\{\mathrm{i}\,k_1'\cdot x_2\}\;\langle\, \phi^{(-)}(x_1')\phi^{(-)}(x_1)a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0
\end{aligned}
$$

Again, the first two lines in the right hand side of the above equality does vanish and going on with the process of reduction we obtain

$$
\begin{aligned}
&\langle\, a(k_1')\,:\phi(x_1')\phi(x_1)\phi(x_2)\,: a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \\
=\;& \exp\{\mathrm{i}\,k_1'\cdot x_1'\}\;\langle\, \phi^{(-)}(x_1)a^\dagger(k_2)\phi^{(-)}(x_2)a^\dagger(k_1)\,\rangle_0 \;+\; k_1 \leftrightarrow k_2 \\
+\;& \text{cyclic permutations of}\quad x_1', x_1, x_2 \\
=\;& \exp\{\mathrm{i}\,k_1'\cdot x_1' - \mathrm{i}\,k_1\cdot x_1 - \mathrm{i}\,k_2\cdot x_2\} \;+\; k_1 \leftrightarrow k_2 \\
+\;& \text{cyclic permutations of}\quad x_1', x_1, x_2 \qquad\quad (\,3!\ \text{terms}\,)
\end{aligned}
$$

Turning now to the evaluation of the quantity

$$
\langle\, a(k_1')a(k_2')\,:\phi(x_2')\phi(x_1')\phi(x_1)\phi(x_2)\,: a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0
$$

the iteration of the above described process of reduction clearly shows that the only non-vanishing contributions read

$$
\begin{aligned}
&\langle\, a(k_1')a(k_2')\,:\phi(x_2')\phi(x_1')\phi(x_1)\phi(x_2)\,: a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \\
=\;& \langle\, a(k_1')a(k_2')\,:\phi^{(+)}(x_2')\phi^{(+)}(x_1')\phi^{(-)}(x_1)\phi^{(-)}(x_2)\,: a^\dagger(k_2)a^\dagger(k_1)\,\rangle_0 \\
+\;& \text{permutations} \qquad\quad (\,4!\ \text{terms}\,) \\
=\;& \exp\{\mathrm{i}\,k_1'\cdot x_1' + \mathrm{i}\,k_2'\cdot x_2' - \mathrm{i}\,k_1\cdot x_1 - \mathrm{i}\,k_2\cdot x_2\} + \text{permutations}
\end{aligned}
$$

Chapter 3

LSZ collision theory

3.1 Asymptotic states and fields

A quite satisfactory general non-perturbative formulation for the interacting field collision theory has been definitely achieved in a series of fundamental papers appeared between the mid-fifties and sixties : namely,

1. Harry Lehmann, Kurt Symanzik and Wolfhart Zimmermann

   Zur Formulierung quantisierter Feldtheorien

   Il Nuovo Cimento **1** (1955) 205

2. Rudolf Haag

   On quantum field theories

   Det Kongelige Danske Videnskabernes Selskab
   Matematisk-fysiske Meddelelser **29** (1955) nr. 12, 1-37

   Quantum field theories with composite particles and asymptotic conditions

   The Physical Review **112** (1958) 669

3. David Ruelle

   On the asymptotic condition in quantum field theory

   Helvetica Physica Acta **35** (1962) 147

Here below I want to merely summarize without proofs the result known as the Haag-Ruelle theorem. First I have to specify the hypotheses and let me consider, without loss of generality, the simplest case of a self-interacting real massive scalar field of mass m. All that will be stated further on can be suitably generalized *mutatis mutandis* to any interacting complex scalar, spinor and vector fields. The generalization of the Haag-Ruelle construction to the massless case is non-trivial and only partially achieved, for it requires a deeper understanding of the infrared divergence problem and the confinement problem in the non-abelian gauge theories.

Consider a real scalar field operator valued tempered distribution $\phi\,(x)$ on the four dimensional Minkowski space-time. We shall suppose the following properties to be indeed verified.

- A unique and normalized vacuum state $|\,0\,\rangle$ exists which satisfies

  $$P^{\,\mu}\,|\,0\,\rangle = 0 \qquad M^{\,\mu\nu}\,|\,0\,\rangle = 0 \qquad \langle\,0\,|\,0\,\rangle = 1$$

  i.e. it is the eigenstate with a null eigenvalue of the energy-momentum and angular momentum self-adjoint operators.

  The vacuum state is postulated to be cyclic, which is nothing but the natural requirement that every state in a quantum field theory has to be obtained in terms of fields. To understand the meaning of cyclicity, let me recall that the (self-interacting) real scalar field $\phi\,(x)$ is not an operator by itself though an operator valued tempered distribution, which becomes a linear operator after smearing with a test-function belonging to the functional space $\mathcal{S}\,(\mathbb{R}^{\,4})$ of the infinitely differentiable rapidly decresing functions

  $$\phi\,(f) = \int \mathrm{d}x\;\phi\,(x)\,f(x)$$

  We shall denote by $\mathfrak{P}\,[\,\phi\,(f)\,]$ the polynomial algebra generated by the operators of the form

  $$\int \mathrm{d}x_1\,\ldots \int \mathrm{d}x_n\;\phi\,(x_1)\,\ldots\,\phi\,(x_n)\;f\,(x_1,\ldots,x_n)$$

  with $f \in \mathcal{S}\,(\mathbb{R}^{\,4n})$. Thus, if we denote by V the set of states of the form $\mathfrak{P}\,[\,\phi\,(f)\,]\,|\,0\,\rangle$, then the Hilbert space is given by the closure $\mathfrak{H} = \overline{V}$.

- The real scalar field operator valued tempered distribution carries a unitary irreducible representation of the Poincaré group

  $$\phi'\,(x) \equiv U\,(a,\omega)\;\phi\,(x)\,U^{\,\dagger}\,(a,\omega) = \phi\,(x^{\,\prime}\,)$$

  $$U\,(a,\omega) = \exp\,\{-\,\mathrm{i}\,a^{\,\mu}\,P_{\,\mu} + \,\mathrm{i}\omega^{\,\rho\sigma}\,M_{\,\rho\sigma}\}$$

  $$x^{\,\prime} = \Lambda\,(\omega)\cdot x + a$$

- The spectrum of the Casimir spin operator $C_s = W^2 = W^\mu W_\mu$ consists in the null eigenvalue, corresponding to a spinless scalar field, while the spectrum of the mass operator $C_m = P^2 = P^\mu P_\mu$ involves the isolated points $P^2 = 0$ belonging to the vacuum and $P^2 = m^2$ belonging to the 1-particle states, as well as the continuum for $P^2 \geq 4m^2$

- The interacting real scalar field fulfills the microcausality property

$$[\,\phi(x)\,,\,\phi(y)\,] = 0 \qquad\qquad (x-y)^2 < 0$$

- There exists a complete set of stable 1-particle improper states $|\,k\,\rangle$ wich satisfy

$$\begin{aligned} \langle\, h\,|\,k\,\rangle &= (2\pi)^3\, 2\omega_{\mathbf{k}}\, \delta\,(\mathbf{h}-\mathbf{k}) &(3.1)\\ k^0 = \omega_{\mathbf{k}} &= \sqrt{\mathbf{k}^2 + m^2} \\ \int Dk\ |\,k\,\rangle\langle\,k\,| &= \mathbf{I}_1 &(3.2)\end{aligned}$$

where the Lorentz covariant measure is defined as usual by

$$\int Dk \ \stackrel{\rm def}{=}\ \int \frac{d\mathbf{k}}{(2\pi)^3\, 2\omega_{\mathbf{k}}} = \frac{1}{(2\pi)^3}\int dk\ \theta\,(k_0)\,\delta\left(k^2 - m^2\right)$$

The proper 1-particle states can be easily constructed by means of

$$|\,\Psi_1\,\rangle = \int Dk\ \Psi\,(k)\,|\,k\,\rangle$$

in such a manner that, for instance,

$$\begin{aligned}\langle\,\Psi_1\,|\,\Psi_1\,\rangle &= \int Dh \int Dk\ \Psi^*\,(h)\,\langle\,h\,|\,k\,\rangle\,\Psi\,(k)\\ &= \frac{1}{(2\pi)^3}\int dk\ \theta(k_0)\,\delta\left(k^2-m^2\right)\ |\,\Psi\,(k)\,|^2\\ &\equiv\ \|\ \Psi_1\ \|\ =\ 1\end{aligned}$$

and consequently

$$P^\mu P_\mu\,|\,\Psi_1\,\rangle = m^2\,|\,\Psi_1\,\rangle$$
$$\langle\,0\,|\,\phi(x)\,|\,\Psi_1\,\rangle\ \neq\ 0$$

Then, if all the above hypoteses do occur, it can be proved, by essentially making use of the Riemann-Lebesgue lemma, that the *asymptotic states and* *fields* indeed exist and are constructed as follows. Let us consider the Fourier transform of the self-interacting real scalar field

$$\phi(x) = \frac{1}{(2\pi)^4} \int dk\ \widetilde{\phi}(k)\ \exp\{-\mathrm{i}k\cdot x\}$$

and define the suitable integral transform

$$\phi(t;f) = \frac{1}{(2\pi)^4} \int dk\ \widetilde{\phi}(-k)\,\widetilde{f}(k)\ \exp\{-\mathrm{i}t\,(k_0-\omega_{\mathbf{k}})\} \tag{3.3}$$

in which

$$\operatorname{supp}\widetilde{f} \subset \left\{k_0>0\,,\,(m-\mu)^2 \le k^2 \le (m+\mu)^2\ ,\ 0<\mu<m\right\}$$

Notice that this latter requirement can be satisfied only in the presence of a mass gap. Consider now the proper multi-particle state

$$|\,\phi_n(f_1,f_2\,,\ldots,f_n\,;\,t)\,\rangle \ \stackrel{\mathrm{def}}{=}\ \prod_{i=1}^{n}\ \phi(t;f_i)\ |\,0\,\rangle$$

Then, eventually, the Haag-Ruelle theorem guarantees the existence of the asymptotic in and out states and fields : namely,

$$\mathrm{s}-\lim{}_{|t|\to\infty}\ |\,\phi_n(f_1,f_2\,,\ldots,f_n\,;\,t)\,\rangle = \prod_{i=1}^{n}\ \phi_{\,\mathrm{as}}(f_i)\ |\,0\,\rangle \tag{3.4}$$

$$f(x) = \int Dk\ \widetilde{f}(\mathbf{k})\ e^{-\mathrm{i}kx} \qquad\qquad \widetilde{f}(\mathbf{k})\ \equiv\ \widetilde{f}\,(k_0=\omega_{\mathbf{k}}\,,\,\mathbf{k})$$

$$\phi_{\,\mathrm{as}}(f)\ \equiv\ \int d\mathbf{x}\ \phi_{\,\mathrm{as}}(x)\,\mathrm{i}\overleftrightarrow{\partial}_0\,f(x)$$
$$\left(\Box+m^2\right)f(x) = 0 = \left(\Box+m^2\right)\phi_{\,\mathrm{as}}(x)$$

$$\phi_{\,\mathrm{as}}(x) = \int Dk\ \left[a_{\,\mathrm{as}}(k)\,e^{-\mathrm{i}kx} + a^{\dagger}_{\,\mathrm{as}}(k)\,e^{\mathrm{i}kx}\right] \tag{3.5}$$

$$\left[\,a_{\,\mathrm{as}}(k)\,,\,a^{\dagger}_{\,\mathrm{as}}(k')\,\right] = (2\pi)^3\,2\omega_{\mathbf{k}}\,\delta\,(\mathbf{k}-\mathbf{k}')$$
$$[\,a_{\,\mathrm{as}}(k)\,,\,a_{\,\mathrm{as}}(k')\,] = 0 = \left[\,a^{\dagger}_{\,\mathrm{as}}(k)\,,\,a^{\dagger}_{\,\mathrm{as}}(k')\,\right]$$

where, as usual, the suffix (as) means (in) for $t\to-\infty$ and (out) for $t\to+\infty$. This is the content of the Haag-Ruelle theorem on the scattering theory.

3.2 LSZ asymptotic theory

The Haag-Ruelle scattering theory, as we have briefly summarized above, is a general, rigorous and non-perturbative framework, which represents a guideline to provide a *rationale* in order to interpret the results of the scattering experiments within the realm of the quantum field theory. To this concern, it turns out that, still nowadays, the problem of constructing exact solutions of the coupled non-linear equations of motion for interacting fields has so far proved too formidable for solution. By the way, the only concrete tool [1] that we have at our disposal to describe interaction in the quantum field theory is the covariant perturbation theory.

In particular, the perturbatively renormalizable interacting quantum field theories do represent the mathematical model which allow us to compute and to predict, up to a very high degree of accuracy, the results of the complicated scattering experiments that take place in the huge modern colliders such as the Large Hadron Collider (LHC) at CERN.

Nonetheless, it is now my aim to try to describe another general non-perturbative formulation for collision theory, beyond the framework of the Haag-Ruelle scattering theory, which represents a cornerstone of modern quantum field theory : the Lehmann-Symanzik-Zimmermann asymptotic theory.

As a first step, it is covenient to set up a general formalism to describe the asymptotic fields, which fulfill free field equations, and the corresponding wave packets and wave functions for quantum fields of arbitrary mass, spin and internal quantum numbers. As a matter of fact, on the ground of the Haag-Ruelle theorem, we shall assume the existence of the asymptotic fields and states for each elementary – *i.e.* not composed by sub-constituents or preons – interacting field of any mass, spin and internal quantum numbers. Actually I shall generally write a covariant normal mode decomposition

$$\Phi_{\mathrm{as}}(x) = \int Dk \left[A_{\mathrm{as},\sigma}(k)\, u_\sigma(k)\, e^{-\mathrm{i}kx} + B^\dagger_{\mathrm{as},\sigma}(k)\, v_\sigma(k)\, e^{\mathrm{i}kx} \right]$$

$$\Phi^\dagger_{\mathrm{as}}(x) = \int Dk \left[B_{\mathrm{as},\sigma}(k)\, v^*_\sigma(k)\, e^{-\mathrm{i}kx} + A^\dagger_{\mathrm{as},\sigma}(k)\, u^*_\sigma(k)\, e^{\mathrm{i}kx} \right]$$

[1] There exist other approximate methods, such as the lattice approach, to treat the interacting quantum fields. However, the latter ones are far less developed and accurate in respect to the covariant perturbation theory.

in which we have set as usual

$$\int Dk = \frac{1}{(2\pi)^3}\int dk\ \theta\,(k_0)\ \delta\left(k^2 - m^2\right) = \frac{1}{(2\pi)^3}\int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}}$$

where the inner quantum number $\sigma = 1, 2, \ldots, n$ will contain all the indices concerning Lorentz as well as internal symmetry group representations, *i.e.* mass, spin, charge, *et cetera*. Furthermore, covariant canonical commutation relations $(-)$ or canonical anticommutation relations $(+)$ are assumed to hold true: namely,

$$[\,A_{\mathrm{as}\,,\sigma}\,(k)\,,\ A^{\dagger}_{\mathrm{as}\,,\sigma'}\,(k'\,)\,]_{\pm} = (2\pi)^3\,2\omega_{\mathbf{k}}\,\delta_{\sigma\sigma'}\,\delta\,(\mathbf{k}-\mathbf{k}'\,) \tag{3.6}$$

$$[\,B_{\mathrm{as}\,,\sigma}\,(k)\,,\ B^{\dagger}_{\mathrm{as}\,,\sigma'}\,(k'\,)\,]_{\pm} = (2\pi)^3\,2\omega_{\mathbf{k}}\,\delta_{\sigma\sigma'}\,\delta\,(\mathbf{k}-\mathbf{k}'\,) \tag{3.7}$$

all the other commutators or anticommutators being equal to zero.

Notice that the polarization index is absent for scalar fields, whilst it takes two possible values for spinor and massless gauge fields though three values for massive vector fields. Then, the smeared and normalizable 1–particle unpolarized states of a general wave field are defined by

$$|\,f\ \mathrm{as}\,\rangle = A^{\dagger}_{\mathrm{as}}\,(f)\,|\,0\,\rangle = \int Dk\ f_{\sigma}\,(k)\,A^{\dagger}_{\mathrm{as}\,,\sigma}\,(k)\,|\,0\,\rangle \tag{3.8}$$

$$|\,g\ \mathrm{as}\,\rangle = B^{\dagger}_{\mathrm{as}}\,(g)\,|\,0\,\rangle = \int Dk\ g_{\sigma}\,(k)\,B^{\dagger}_{\mathrm{as}\,,\sigma}\,(k)\,|\,0\,\rangle \tag{3.9}$$

$$\langle\,0\,|\,0\,\rangle = \langle\,f\ \mathrm{as}\,|\,f\ \mathrm{as}\,\rangle = \langle\,g\ \mathrm{as}\,|\,g\ \mathrm{as}\,\rangle = 1 \tag{3.10}$$

The normalizable wave functions read

$$\begin{aligned}
f_{\mathrm{as}}\,(x) &= \langle\,0\,|\,\Phi_{\mathrm{as}}\,(x)\,|\,f\,\rangle \\
&= \Big\langle\,\big[\,\Phi_{\mathrm{as}}\,(x)\,,\ A^{\dagger}_{\mathrm{as}}\,(f)\,\big]_{\pm}\Big\rangle_0 \\
&= \int Dk\ f_{\sigma}\,(k)\,u_{\sigma}\,(k)\,e^{-\,\mathrm{i}\,k\,x} \\
g_{\mathrm{as}}\,(x) &= \langle\,0\,|\,\Phi^{\dagger}_{\mathrm{as}}\,(x)\,|\,g\,\rangle \\
&= \Big\langle\,\big[\,\Phi^{\dagger}_{\mathrm{as}}\,(x)\,,\ B^{\dagger}_{\mathrm{as}}\,(g)\,\big]_{\pm}\Big\rangle_0 \\
&= \int Dk\ g_{\sigma}\,(k)\,\upsilon^{*}_{\sigma}\,(k)\,e^{-\,\mathrm{i}\,k\,x}
\end{aligned}$$

In the case of several different particles and antiparticles it is necessary to consider the states

$$|\,f_1\,f_2\ \ldots\ f_M\,;\,g_1\,g_2\ \ldots\ g_N\,\rangle = \prod_{i\,=\,1}^{M}\ \prod_{j\,=\,1}^{N}\ A^{\dagger}_{\mathrm{as}}\,(\,f_i\,)\ B^{\dagger}_{\mathrm{as}}\,(\,g_j\,)\ \,|\,0\,\rangle$$

If the 1-particle wave packets are normalized to unit and if the particles are indeed different – which is necessarily true in the case of anticommuting fields – then the multi-particle states are also normalized to one.

In the case of quantum fields with integer spin, *i.e.* boson fields that satisfy canonical commutation relations, we have to treat separately the case in which among the inner quantum numbers $\sigma_1, \sigma_2, \ldots, \sigma_N$ there are some identical ones, *i.e.* when among the N particles there are several groups of identical particles. In the case of several groups $\nu_1, \nu_2, \ldots, \nu_a$ of identical particles we shall correspondingly obtain

$$|\, f_1 f_2 \ldots f_M \,\rangle = \Big(\prod [\, \nu\, !\,] \Big)^{-\frac{1}{2}} \prod_{i=1}^{M} A_{\mathrm{as}}^{\dagger}\,(\, f_i\,)\,|\,0\,\rangle \tag{3.11}$$

where the following notation has been used

$$\prod [\, \nu\, !\,] \;=\; \nu_1!\,\nu_2!\,\cdots\,\nu_a! \tag{3.12}$$

and a quite analogous formula evidently holds true for antiparticles.

In order to discuss the general framework for a perturbative approach to the scattering theory of interacting quantum fields, I can restrict myself to the simplest case : the self-interacting real scalar field. This is because, on the one hand, I want to deal with the minimal degrees of freedom content and to avoid thereby the proliferation of indices, coupling parameters, field operators *et cetera.* On the other hand, the description of the scattering theory for a set of interacting quantum fields with higher spin and any further internal quantum numbers, does not introduce any complications of principle, but more or less heavy technicalities to face with.

Admittedly, the self-interacting real scalar quantum field theory does not help to describe any realistic process within the context of high energy physics and elementary particle physics. Conversely, its euclidean version does truly underlie the description of many condensed matter systems near their critical points, *i.e.* the so called euclidean $\lambda\,\phi^4$ theory is the cornerstone to describe the second order phase transitions for many different physical systems – to this concern, see the excellent textbook :

J. J. Binney, N. J. Dowrick, A. J. Fisher and M. E. J. Newman
The theory of critical phenomena
An introduction to the renormalization group
Clarendon Press, Oxford (UK) 1995

Last but not least, taking precisely inspiration from the classical Landau theory of the second order phase transitions

Lev Davidovic Landau, Phys. Zs. Sowjet **11** (1937) 545

L. D. Landau & E. M. Lifchitz (1967) *Physique Statistique*, MIR, Moscou some particular version of the self-interacting real scalar quantum field theory has been used to implement the spontaneous symmetry breaking mechanism in perturbation theory :

Jeoffrey Goldstone
Field Theories with Superconductor Solutions
Il Nuovo Cimento **19** (1961) 154

Jeoffrey Goldstone, Abdus Salam & Steven Weinberg
Broken Symmetries
The Physical Review **127** (1962) 965-970

François Englert & Robert Brout
The Physical Review Letters **13** (1964) 321

Peter Ware Higgs
Physics Letters **12** (1964) 132
The Physical Review Letters **13** (1964) 508

G.S. Guralnik, C.R. Hagen & T.W.B. Kibble
The Physical Review Letters **13** (1964) 585

It turns out that the so named Higgs boson H^0, an hypotetic elementary scalar massive particle, the only one in the Standard Model which has not yet been observed, is the key ingredient to produce the spontaneous breaking of the $SU(2)$ flavour gauge symmetry, by means of which the masses of the gauge bosons $W^{\pm}$ and Z^0 are generated in perturbation theory.

Consider the classical Lagrange density

$$\mathcal{L} = \frac{1}{2}\, g^{\mu\nu}\, \partial_\mu \phi(x)\, \partial_\nu \phi(x) - \frac{1}{2}\, m^2\, \phi^2(x) - \frac{\lambda}{4!}\, \phi^4(x)$$

leading to the conjugate momentum field

$$\Pi(x) = \dot{\phi}(x)$$

and to the classical hamiltonian functional

$$\begin{aligned} H &= H_0 + H_{\rm I} \geq 0 \\ H_0 &= \frac{1}{2}\int d\mathbf{x}\left[\,\Pi^2(x) - \phi(x)\boldsymbol{\nabla}^2\phi(x) + m^2\,\phi^2(x)\,\right] \\ H_{\rm I} &= \int d\mathbf{x}\; \frac{\lambda}{4!}\, \phi^4(x) \end{aligned}$$

The total energy-momentum and orbital angular momentum of the classical self-interacting real scalar field can be readily obtained from Noether theorem and read

$$P_0 \equiv H \qquad \mathbf{P} \equiv \int d\mathbf{x}\; \Pi(x)\, \boldsymbol{\nabla}\phi(x)$$

$$L_{\mu\nu} = \int d\mathbf{x}\; \Big[\, x_{\nu}\, T_{0\mu}(t,\mathbf{x}) - x_{\mu}\, T_{0\nu}(t,\mathbf{x}) \Big]$$

The classical Euler-lagrange wave field equations are

$$\left(\Box + m^2\right)\phi(x) \;=\; -\,\frac{\lambda}{3!}\,\phi^3(x)$$

and a formal implicit general solutions of the interacting field equations can be cast in the so named Yang-Feldman form : namely,

$$\phi(x) = \begin{cases} \sqrt{Z}\,\phi_{\text{in}}(x) - (\lambda/3!)\int dy\; D_{\text{ret}}(x-y\,;\,m)\,\phi^3(y) \\ \sqrt{Z}\,\phi_{\text{out}}(x) - (\lambda/3!)\int dy\; D_{\text{adv}}(x-y\,;\,m)\,\phi^3(y) \end{cases}$$

where $Z > 0$ is an arbitrary constant whereas

$$\left(\Box + m^2\right)\phi_{\text{as}}(x) = 0$$

$$D_{\text{ret}}(x\,;\,m) = \frac{1}{(2\pi)^4}\int dk\; \frac{\exp\{-\mathrm{i}k\cdot x\}}{m^2 - k^2 - 2i\varepsilon k_0}$$

$$D_{\text{adv}}(x\,;\,m) = \frac{1}{(2\pi)^4}\int dk\; \frac{\exp\{-\mathrm{i}k\cdot x\}}{m^2 - k^2 + 2i\varepsilon k_0}$$

with the manifest properties

$$\begin{cases} D_{\text{ret}}(x\,;\,m) = 0 & \text{for } x_0 < 0 \\ D_{\text{adv}}(x\,;\,m) = 0 & \text{for } x_0 > 0 \end{cases}$$

$$\left(\Box + m^2\right) D_{\text{ret}}(x\,;\,m) = \delta(x) = \left(\Box + m^2\right) D_{\text{adv}}(x\,;\,m)$$

From the above properties of the retarded and advanced Green's functions, it is clear that for any classical field function that falls down to zero when $|\,t\,| \to \infty$ we shall find the asymptotic behaviour

$$\phi(x) \;\stackrel{|\,t\,|\to\infty}{\sim}\; \sqrt{Z}\,\phi_{\text{as}}(x)$$

The transition to the quantum field theory is formally achieved after imposing the canonical equal-time commutation relations

$$[\,\phi(t,\mathbf{x})\,,\,\phi(t,\mathbf{y})\,] = 0 = [\,\Pi(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\,]$$
$$[\,\phi(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\,] \;=\; i\,\delta(\mathbf{x}-\mathbf{y})$$

and the Euler-Lagrange field equations of motion will formally be turned into Heisenberg operator equations

$$\dot{\phi}(t,\mathbf{x}) = \frac{1}{\mathrm{i}}\,[\,\phi(t,\mathbf{x})\,,\,H\,] = \Pi(t,\mathbf{x})$$

$$\dot{\Pi}(t,\mathbf{x}) = \frac{1}{\mathrm{i}}\,[\,\Pi(t,\mathbf{x})\,,\,H\,] = \left(\boldsymbol{\nabla}^2 - m^2\right)\phi(t,\mathbf{x}) - \frac{\lambda}{3!}\,\phi^3(t,\mathbf{x})$$

However, the latter formal developments do indeed suffer from a serious pathology, which relies on the fact that the product of field operators at the same space-time point turns out to be unavoidably ill-defined. Actually, I have already noticed that the interacting fields are not operator, but merely operator valued tempered distributions, *i.e.* they become operators only after a smearing with test functions belonging to the functional space $\mathcal{S}(\mathbb{R}^4)$.

Hence, I will formulate the scattering theory in the covariant perturbative approach for a real self-interacting scalar quantized field, in accordance with the so named Lehmann-Symanzik-Zimmermann (LSZ) basic assumptions.

1. The self-interacting real scalar field is an operator valued tempered distribution that carries a unitary irreducible representation of the Poincaré group

$$\phi'(x) \equiv U(a,\omega)\,\phi(x)U^{\dagger}(a,\omega) = \phi(x')$$

$$U(a,\omega) = \exp\{-\,\mathrm{i}a^{\mu}P_{\mu} + \mathrm{i}\omega^{\rho\sigma}M_{\rho\sigma}\}$$

where $x'^{\mu} = [\,\Lambda(\omega)\,]^{\mu}{}_{\nu}\,x^{\nu} + a^{\mu}$

2. A unique and cyclic (normalized) vacuum state $|\,0\,\rangle$ exists which does fulfill

$$P^{\mu}\,|\,0\,\rangle = 0 \qquad M^{\mu\nu}\,|\,0\,\rangle = 0 \qquad \langle\,0\,|\,0\,\rangle = 1$$

3. The self-interacting real scalar field becomes a linear operator after smearing with a test-function belonging to the functional space $\mathcal{S}(\mathbb{R}^4)$ of the infinitely differentiable rapidly decreasing functions

$$\phi(f) = \int \mathrm{d}x\ \phi(x)\,f(x)$$

We shall denote by $\mathfrak{P}[\,\phi(f)\,]$ the polynomial algebra generated by the operators of the form

$$\int \mathrm{d}x_1 \ldots \int \mathrm{d}x_n\ \phi(x_1)\ldots\phi(x_n)\ f(x_1,\ldots,x_n)$$

with $f \in \mathcal{S}(\mathbb{R}^{4n})$. Thus, if we denote by V the set of states of the form $\mathfrak{P}[\,\phi(f)\,]\,|\,0\,\rangle$, then the Hilbert space is given by the closure $\mathfrak{H} = \overline{V}$.

4. The field equations for the operator valued tempered distribution which describe the self-interacting real scalar field can be written in the so named modified Yang-Feldman integral form

$$\begin{aligned}\phi(x) &= \sqrt{Z}\,\phi_{\mathrm{in}}(x) + \int dy\; D_{\mathrm{ret}}(x-y)\,g(y)\,\mathcal{K}_y\,\phi(y) \\ &= \sqrt{Z}\,\phi_{\mathrm{out}}(x) + \int dy\; D_{\mathrm{adv}}(x-y)\,g(y)\,\mathcal{K}_y\,\phi(y) \qquad (3.13)\end{aligned}$$

where $Z > 0$ is an arbitrary constant, $\mathcal{K}_x \equiv (\Box_x + m^2)$ is the kinetic differential operator of the free field theory, whereas the function $g(x)$ with values in the range $(0,1)$ does represent the extent of switching on the interaction. In those regions where $g(x) = 0$ the interaction is absent, in those regions where $g(x) = 1$ it is switched on completely, while for $0 < g(x) < 1$ the interaction is switched on only partially. In other words, by replacing the classical interaction term

$$-\frac{\lambda}{3!}\,\phi^3(x) = K_x\,\phi(x)$$

in the equations of motion by

$$g(x)\,\mathcal{K}_x\,\phi(x)$$

we obtain at the quantum level [2] an interaction switched on with an intensity $g(x)$

5. For any rapidly decreasing Klein-Gordon wave packet

$$f \in \mathcal{S}\left(\mathbb{R}^4\right) \qquad \mathcal{K}_x\,f(x) = \left(\Box_x + m^2\right)\,f(x) = 0$$

let us define the field operator $\phi(t; f)$ by smearing $\phi(x)$ over a space-like region in accordance to

$$\phi(t; f) = \mathrm{i}\int d\mathbf{x}\; f^*(x)\,\overleftrightarrow{\partial}_0\,\phi(x)$$

Then, for any pair of proper states $\alpha\,,\,\beta \in \mathfrak{H}$, the LSZ asymptotic condition states :

$$\lim_{|t|\to\infty}\;\langle\,\alpha\,|\,\phi(t; f)\;|\,\beta\,\rangle = \sqrt{Z}\,\langle\,\alpha\,|\,\phi_{\mathrm{as}}(f)\,|\,\beta\,\rangle \qquad \forall\,\alpha\,,\,\beta \in \mathfrak{H}$$

[2] To this concern see also N.N. Bogoliubov & D.V. Shirkov, *Introduction to the Theory of Quantized Fields*, Interscience Publishers, New York, 1959, §**17** pp. 197-204

where, as customary,

$$\mathcal{K}_x\,\phi_{\text{as}}(x) \;=\; \left(\Box_x + m^2\right)\,\phi_{\text{as}}(x) \;=\; 0$$

$$\phi_{\text{as}}(f) \;\equiv\; \int d\mathbf{x}\;\phi_{\text{as}}(x)\,\mathrm{i}\overleftrightarrow{\partial}_0\,f(x) \;=\; \int Dk\;\widetilde{f}(k)\;a^{\dagger}_{\text{as}}(k)$$

$$\int Dk \;\stackrel{\text{def}}{=}\; \int \frac{d\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}} = \frac{1}{(2\pi)^3}\int dk\;\theta(k_0)\,\delta\left(k^2-m^2\right)$$

$$f(x) = \int Dk\;\widetilde{f}(k)\;e^{-\,\mathrm{i}\,k\,x}$$

$$\parallel f \parallel = \int d\mathbf{x}\;f^{*}(x)\,\mathrm{i}\overleftrightarrow{\partial}_0 f(x) \;=\; \int D\,k\;|\widetilde{f}(\mathbf{k})|^2 \;=\; 1$$

$$\phi_{\text{as}}(x) = \int Dk\;\left[\,a_{\text{as}}(k)\,e^{-\,\mathrm{i}\,k\,x} + a^{\dagger}_{\text{as}}(k)\,e^{\,\mathrm{i}\,k\,x}\right]$$

$$\left[\,a_{\text{as}}(k)\,,\,a^{\dagger}_{\text{as}}(k')\,\right] = (2\pi)^3\,2\omega_{\mathbf{k}}\,\delta\,(\mathbf{k}-\mathbf{k}')$$

$$[\,a_{\text{as}}(k)\,,\,a_{\text{as}}(k')\,] = 0 = \left[\,a^{\dagger}_{\text{as}}(k)\,,\,a^{\dagger}_{\text{as}}(k')\,\right]$$

$$[\,\phi_{\text{as}}(x)\,,\,\phi_{\text{as}}(y)\,] = \frac{1}{i}\,D\,(x-y) \qquad\qquad (\text{Pauli} - \text{Jordan distribution})$$

Notice the canonical engeneering dimensions in natural units :

$$[\,f\,] = \text{eV} = [\,\phi_{\text{as}}\,] \qquad [\,\widetilde{f}\,] = \text{cm} = [\,a_{\text{as}}\,] = \qquad [\,D\,k\,] = \text{eV}^2$$

6. From the conventional definitions

$$a_{\text{as}}(k')\,|\,0\text{ as}\,\rangle = 0 = \langle\,0\text{ as}\,|\,a^{\dagger}_{\text{as}}(k)$$

with the standard normalization

$$\langle\,0\text{ as}\,|\,0\text{ as}\,\rangle = \langle\,0\,|\,0\,\rangle \;=\; 1 \tag{3.14}$$

together with

$$\mathfrak{H}_{\text{as}} = \overline{V}_{\text{as}} \qquad\qquad \overline{V}_{\text{as}} = \mathfrak{P}\,[\,\phi_{\text{as}}(\,f\,)\,]\,|\,0\text{ as}\,\rangle$$

the *asymptotic completeness* is supposed to be true

$$\mathfrak{H}_{\text{in}} = \mathfrak{H} = \mathfrak{H}_{\text{out}}$$

which means the absence of the bound states in perturbation theory.

Let me conclude with a philosophical comment. In quantum field theory a Fock' space of states is generated from a unique vacuum $|\,0\ \text{in}\,\rangle$ by a free field operator valued tempered distribution denoted by $\phi_{\,\text{in}}\,(x)$. This is the stage upon which the whole dynamical process takes place, *i.e.* preparation of the beam and target states in a laboratory collision experiment. Then, to the best of our knowledges, all the physical observables, that means the probabilities associated to the scattering amplitudes, must be expressed in terms of that unique free field $\phi_{\,\text{in}}\,(x)$ and the dimensionaless small coupling $0 < \lambda < 1$, which characterize the real scalar field self-interaction. In particular, this appears to be the case for the interacting field $\phi\,(x)$. Intuitively, we can imagine the relation between these two fields as follows : in the remote past $\phi_{\,\text{in}}\,(x)$ is some suitable limit of $\phi\,(x)$. This refers, of course, to some definite laboratory collision experiment under consideration and actually applies only when the partecipating colliding particles are well separated from each other. To this concern, it is worthwhile to keep in mind that, typically, elementary particle physics interactions range within $10^{-15} \div 10^{-13}$ cm. In order to implement this idea we may naturally assume that the coupling terms in the equations of motion are affected by some adiabatic cutoff function $g\,(x)$ equal to one at finite times and distances and vanishing smoothly when $|\,x^{\mu}\,| \;\rightarrow\; \infty$. All the physical quantities have to be understood in the limit when this adiabatic swtching is removed. Then the adiabatic hypothesis states that under those assumptions

$$\phi\,(x\,) \;\rightarrow\; Z^{1/2}\,\phi_{\,\text{in}}\,(x) \qquad \text{when } t \;\rightarrow\; -\,\infty$$

This is the physical meaning of the Lehmann-Symanzik-Zimmermann asymptotic assumtion, which appears to be quite natural and reliable when massive fields, repulsive or very weakly attractive interactions are involved in high energy collision processes. For instance, in the case of massless fields things are much more subtle, owing to the onset of infrared singularities or even the confinement mechanism for non-Abelian gauge theories.

A fully satisfactory and exhaustive understanding of these subtleties is still far from being achieved, since the solution lies outside the realm of a perturbative approach. Nonetheless, the LSZ asymptotic assumption and the related reduction formulæ are still the most powerful guideline we have at hand nowadays to depict the scattering experiments in the perturbative approach to the quantum field theory.

3.3 $S-$matrix generating functional

In the perturbative covariant framework, as specified by the above listed LSZ assumptions for the scattering theory of interacting quantum fields, the $S-$matrix is a unitary operator which fulfills the following relationships :

$$S\,:\,\mathfrak{H}_{\,\mathrm{in}}\longrightarrow\mathfrak{H}_{\,\mathrm{out}}\qquad\qquad S^{\dagger}\,:\,\mathfrak{H}_{\,\mathrm{out}}\longrightarrow\mathfrak{H}_{\,\mathrm{in}}$$

$$\langle\,\alpha\ \mathrm{in}\,|\,S=\langle\,\alpha\ \mathrm{out}\,|\qquad\qquad S^{\dagger}\,|\,\beta\ \mathrm{in}\,\rangle=|\,\beta\ \mathrm{out}\,\rangle$$

$$\phi_{\,\mathrm{in}}\,(x)=S\,\phi_{\,\mathrm{out}}\,(x)\,S^{\dagger}\qquad\qquad\phi_{\,\mathrm{out}}\,(x)=S^{\dagger}\,\phi_{\,\mathrm{in}}\,(x)\,S \tag{3.15}$$

$$S\,S^{\dagger}=S^{\dagger}\,S=\mathbf{I}\qquad\qquad(\,\text{unitarity conditions}\,)$$

$$U\,(a\,,\,\omega)\,S\,U^{\dagger}\,(a\,,\,\omega)=S\qquad\qquad(\,\text{Poincaré covariance}\,)$$

$$\langle\,\alpha\ \mathrm{out}\,|\,\beta\ \mathrm{in}\,\rangle=\langle\,\alpha\ \mathrm{in}\,|\,S\,|\,\beta\ \mathrm{in}\,\rangle=\langle\,\alpha\ \mathrm{out}\,|\,S\,|\,\beta\ \mathrm{out}\,\rangle$$

the latter relation just yielding the *probability amplitude* of a scattering experiment.

The chronologically ordered exponential operator for the self-interacting real scalar field is defined by

$$\begin{aligned} T\,[\,J\,] &\stackrel{\mathrm{def}}{=} T\ \exp\{\,i\,\phi\,(J)\} \\ &= T\ \exp\left\{\frac{i}{\hbar}\int_{-\infty}^{\infty}dx_0\int d\mathbf{x}\ \phi\,(x)\,J(x)\right\} \end{aligned} \tag{3.16}$$

and thereby the generating functional for the Green's functions, that is the vacuum expectation values of the chronologically ordered products of fields at different space-time points, will be defined in the usual way as in the free field case : namely,

$$\begin{aligned} Z\,[\,J\,] &\stackrel{\mathrm{def}}{=} \langle\,0\,|\,T\,[\,J\,]\,|\,0\,\rangle=\sum_{n=0}^{\infty}\frac{i^{\,n}}{n!}\int\mathrm{d}x_{\,1}\ldots\int\mathrm{d}x_{\,n} \\ &\times\ J(x_1)\ldots\,J(x_n)\ \langle\,0\,|\,T\phi\,(x_1)\ldots\,\phi\,(x_{\,n})\,|\,0\,\rangle \end{aligned} \tag{3.17}$$

where the classical external sources $J(x_i)$ $(i=1,2,\ldots,n)$ can always be supposed to be infinitely differentiable and rapidly decreasing functions in $\mathcal{S}\,(\mathbb{R}^{4n})$. Taking one functional derivative we get

$$\frac{\delta\,T\,[\,J\,]}{i\,\delta\,J(x)}=T\,\phi\,(x)\ \exp\{\,i\,\phi\,(J)\}=T\ \exp\{\,i\,\phi\,(J)\}\ \phi\,(x)$$

and from the Yang-Feldman form (3.13) of the equations of motion we come to the equalities

$$\begin{aligned}
\frac{\delta T[J]}{i\,\delta J(x)} &= \sqrt{Z}\,T[J]\,\phi_{\rm in}(x) \\
&+ \int dy\; D_{\rm ret}(x-y)\,g(y)\,\mathcal{K}_y\,\Big(\delta T[J]/i\,\delta J(y)\Big) \\
&= \sqrt{Z}\,\phi_{\rm out}(x)\,T[J] \\
&+ \int dy\; D_{\rm adv}(x-y)\,g(y)\,\mathcal{K}_y\,\Big(\delta T[J]/i\,\delta J(y)\Big)
\end{aligned}$$

so that

$$\begin{aligned}
&\sqrt{Z}\;(\phi_{\rm out}(x)\,T[J] - T[J]\,\phi_{\rm in}(x)) \\
&= \int dy\; D(x-y)\,g(y)\,\mathcal{K}_y\,\frac{\delta T[J]}{i\,\delta J(y)}
\end{aligned} \tag{3.18}$$

where $D(x) = D_{\rm ret}(x) - D_{\rm adv}(x)$ is the Pauli-Jordan distribution. Then, after multiplication to the left by the $S-$matrix of both members of the last equality we obtain

$$\begin{aligned}
S(g)\,\phi_{\rm out}(x)\,T[J] &- S(g)\,T[J]\,\phi_{\rm in}(x) \\
&= Z^{-1/2}\int dy\; D(x-y)\,g(y)\,\mathcal{K}_y\,\frac{\delta\,S(g)\,T[J]}{i\,\delta J(y)} \\
&= \Big[\phi_{\rm in}(x)\,,\,S(g)\,T[J]\Big]
\end{aligned} \tag{3.19}$$

where I have made use of the relation (3.15) and I have denoted by $S(g)$ the scattering matrix with the interaction switched on with an intensity g. The physical $S-$matrix $S \equiv S(g=1)$ will be obtained by putting $g = 1$ at the very end of the calculations.

The formal solution of the above functional equation (3.19) is provided by

$$S(g)\,T[J] = \mathbb{K}\,Z[J] \tag{3.20}$$

$$\begin{aligned}
\mathbb{K} &= :\exp\left\{Z^{-1/2}\int {\rm d}x\;\phi_{\rm in}(x)\,g(x)\,\mathcal{K}_x\,\frac{\delta}{i\,\delta J(x)}\right\}: \qquad (3.21) \\
&= \sum_{n=0}^{\infty}\frac{(-{\rm i})^n}{n!}\,Z^{-n/2}\int {\rm d}x_1\int {\rm d}x_2\,\ldots\int {\rm d}x_n \\
&\quad :\phi_{\rm in}(x_1)\,\phi_{\rm in}(x_2)\,\ldots\,\phi_{\rm in}(x_n):\;g(x_1)\,g(x_2)\,\ldots\,g(x_n) \\
&\quad \mathcal{K}(x_1)\,\frac{\delta}{\delta J(x_1)}\,\mathcal{K}(x_2)\,\frac{\delta}{\delta J(x_2)}\,\ldots\,\mathcal{K}(x_n)\,\frac{\delta}{\delta J(x_n)}
\end{aligned}$$

where ($\jmath$ repeated but not summed)

$$\mathcal{K}(x_\jmath) = \left(\Box_\jmath + m^2\right) = m^2 + \sum_{\mu,\nu=0}^{3} g^{\mu\nu}\left(\partial^2/\partial x_\jmath^\mu\,\partial x_\jmath^\nu\right)$$

`Proof.` From the Baker-Campbell-Hausdorff formula

$$\exp\{A+B\} = \exp\{A\}\exp\{B\}\exp\left\{-\frac{1}{2}[A,B]\right\}$$

which holds true whenever

$$[A,B] = c - \text{number}$$

we get

$$\left(\exp\int \mathrm{d}x\ \phi_{\text{as}}^{(-)}(x)\,g(x)\right)\left(\exp\int dy\ \phi_{\text{as}}^{(+)}(y)\,g(y)\right)$$
$$= :\ \exp\int dz\ \phi_{\text{as}}(z)\,g(z)\ :$$

which follows directly from the $c-$number commutator

$$\left[\phi_{\text{as}}^{(-)}(x),\,\phi_{\text{as}}^{(+)}(y)\right] = \frac{1}{i}\,D^{(-)}(x-y)$$

and from the definition of the exponentials as Taylor's expansions, taking into account that the normal product is defined as the original product reduced to its normal form with all the commutator functions being taken equal to zero in the process of reduction. Moreover we have the identity

$$[A,\exp\{B\}] = [A,B]\exp\{B\}$$

hence

$$\left[\phi_{\text{as}}(x),\,:\ \exp\int dy\ \phi_{\text{as}}(y)\,g(y)\ :\right]$$
$$= \frac{1}{i}\int dy\ D(x-y)\,g(y)\ :\ \exp\int dz\ \phi_{\text{as}}(z)\,g(z)\ : \qquad (3.22)$$

Since the normalization condition (3.14) gives

$$\langle 0\ \text{in}\,|\,S\,T[J]\,|\,0\ \text{in}\rangle = Z[J]$$

we eventually find

$$S(g) \;=\; \mathbb{K}\, Z[\,J\,]\,\rfloor_{J=0} \tag{3.23}$$

that is

$$\begin{aligned} S(g) \;=\; & \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!}\, Z^{-n/2} \int \mathrm{d}x_1 \int \mathrm{d}x_2 \,\ldots\, \int \mathrm{d}x_n \\ & :\phi_{\mathrm{in}}(x_1)\,\phi_{\mathrm{in}}(x_2)\,\ldots\,\phi_{\mathrm{in}}(x_n):\; g(x_1)\,g(x_2)\,\ldots\,g(x_n) \\ & \mathcal{K}(x_1)\,\mathcal{K}(x_2)\,\ldots\,\mathcal{K}(x_n)\,\langle\, 0\,|\,T\phi(x_1)\,\phi(x_2)\,\ldots\,\phi(x_n)\,|\,0\,\rangle \end{aligned}$$

and in the limit in which the interaction is completely switched on

$$\begin{aligned} S \;=\; & \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!}\, Z^{-n/2} \int \mathrm{d}x_1 \int \mathrm{d}x_2 \,\ldots\, \int \mathrm{d}x_n \\ & :\phi_{\mathrm{in}}(x_1)\,\phi_{\mathrm{in}}(x_2)\,\ldots\,\phi_{\mathrm{in}}(x_n): \\ & \mathcal{K}(x_1)\,\mathcal{K}(x_2)\,\ldots\,\mathcal{K}(x_n) \\ & \langle\, 0\,|\,T\phi(x_1)\,\phi(x_2)\,\ldots\,\phi(x_n)\,|\,0\,\rangle \end{aligned}$$

Q.E.D.

It is not difficult to gather that the above expression for the generating functional of the $S-$matrix can be generalized [3] in a straightforward manner to the case of anticommuting fields and to the case of fields with any spin and internal quantum numbers. However, the corresponding general explicit formulæ will appear to be admittedly rather cumbersome and will thereby be put forward in the sequel only for some specific simple cases of a particular physical interest.

[3] To this concern see *e.g.* the textbook by Claude Itzykson & Jean-Bernard Zuber, *Quantum Field Theory* , McGraw-Hill, New York, 1980, pp. 205-224

3.4 Reduction formulas

Consider now, for instance, the elastic reaction $1+2 \longrightarrow 3+4$ that will be described by the matrix element

$$
\begin{aligned}
&\langle\, f_{4\,,\,\mathrm{out}}\ f_{3\,,\,\mathrm{out}}\,|\,f_{1\,,\,\mathrm{in}}\ f_{2\,,\,\mathrm{in}}\,\rangle = \langle\, f_{4\,,\,\mathrm{in}}\ f_{3\,,\,\mathrm{in}}\,|\,S\,|\,f_{1\,,\,\mathrm{in}}\ f_{2\,,\,\mathrm{in}}\,\rangle \\
= &\int Dk_4\ f_4^{\,*}(k_4)\int Dk_3\ f_3^{\,*}(k_3)\int Dk_2\ f_2(k_2)\int Dk_1\ f_1(k_1) \\
\times &\ \Big\langle\, a_{\,\mathrm{in}}(k_4)\,a_{\,\mathrm{in}}(k_3)\ \mathbb{K}\,Z[\,J\,]\,\rfloor_{J=0}\ a^{\dagger}_{\,\mathrm{in}}(k_2)\,a^{\dagger}_{\,\mathrm{in}}(k_1)\,\Big\rangle_0
\end{aligned}
$$

where the incident and scattered particle wave packets are defined in the usual way, that is

$$
|\,f_{\jmath\,,\,\mathrm{as}}\,\rangle = a^{\dagger}_{\,\mathrm{as}}(f_\jmath)\,|\,0\,\rangle = \int Dk_\jmath\ f_\jmath(k_\jmath)\,a^{\dagger}_{\,\mathrm{as}}(k_\jmath)\,|\,0\,\rangle \qquad (\jmath = 1,2,3,4)
$$

Since the symbol $\mathbb{K}$ contains the normal products of the asymptotic incoming fields $:\ \phi_{\,\mathrm{in}}(x_1)\,\phi_{\,\mathrm{in}}(x_2)\ \ldots\ \phi_{\,\mathrm{in}}(x_n)\ :$ in which

$$
\phi_{\,\mathrm{as}}(x) = \int Dk\ \left[\,a_{\,\mathrm{as}}(k)\,e^{-\,\mathrm{i}\,k\,x} + a^{\dagger}_{\,\mathrm{as}}(k)\,e^{\,\mathrm{i}\,k\,x}\,\right]
$$

a little thought – see Problem 1. – will convince ourselves that the solely surviving non-vanishing term is that one with $n = 4$. More generally, for an elastic scattering process with N incoming identical particles and N' outgoing identical particles

$$
1+2+\ldots+N\ \longmapsto\ 1'+2'+\ldots+N'
$$

we shall evidently obtain the dimensionless amplitude

$$
\begin{aligned}
&\langle\, N'\ \mathrm{in}\,|\,S(g)\,|\,N\ \mathrm{in}\,\rangle\ =\ (\,N\,!\,N'\,!\,)^{-\,1/2}\ \times \\
&\prod_{\jmath=1}^{N'}\int dx'_\jmath\int Dk'_\jmath\ f^{\,*}_\jmath(k'_\jmath)\,\mathrm{i}\,Z^{-\,1/2}\ \exp\{\,\mathrm{i}\,k'_\jmath\cdot x'_\jmath\,\}\ g(x'_\jmath)\,\mathcal{K}(x'_\jmath) \\
&\prod_{\imath=1}^{N}\int dx_\imath\int Dk_\imath\ f_\imath(k_\imath)\,\mathrm{i}\,Z^{-\,1/2}\ \exp\{-\,\mathrm{i}\,k_\imath\cdot x_\imath\,\}\ g(x_\imath)\,\mathcal{K}(x_\imath) \\
&\langle\,0\,|\,T\phi(x_1)\,\ldots\,\phi(x_N)\,\phi(x'_1)\,\ldots\,\phi(x'_{N'})\,|\,0\,\rangle
\end{aligned}
$$

Taking eventually a complete switching on of the interaction, which amounts to set $g(x_\imath) = 1 = g(x'_\jmath)$ at the very end, we can write

$$\langle\, N'\ \mathrm{out}\,|\, N\ \mathrm{in}\,\rangle \;=\; (\, N!\, N'!\,)^{-1/2}$$

$$\prod_{\jmath=1}^{N'} \mathrm{i}\, Z^{-1/2} \int Dk'_\jmath\ f^*_\jmath\,(k'_\jmath)\ \lim_{k'^{\,2}_\jmath \to m^2}\ \left(m^2 - k'^{\,2}_\jmath\right)$$

$$\prod_{\imath=1}^{N} \mathrm{i}\, Z^{-1/2} \int Dk_\imath\ f_\imath(k_\imath)\ \lim_{k^2_\imath \to m^2}\ \left(m^2 - k^2_\imath\right)$$

$$(2\pi)^4\, \delta\,(\, K_i - K'_f\,)\, \widetilde{G}_{N+N'}\,(\,-\,k_1,\,\ldots\,,-\,k_N\,;\,k'_1,\,\ldots\,,k'_{N'}\,) \qquad (3.24)$$

where I have introduced the $n-$point Green's functions in the momentum space : namely,

$$\langle\, 0\,|\, T\,\phi(x_1)\,\phi(x_2)\ \ldots\ \phi(x_n)\,|\,0\,\rangle = \int \frac{\mathrm{d}^4 p_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 p_2}{(2\pi)^4} \cdots \int \frac{\mathrm{d}^4 p_n}{(2\pi)^4}$$

$$(2\pi)^4\, \delta\,(\,P\,)\ \widetilde{G}_n\ (\,p_1, p_2\,\ldots\,, p_n\,) \prod_{\jmath=1}^{n} \exp\{\,-\,\mathrm{i}\, p_\jmath \cdot x_\jmath\} \quad (3.25)$$

where the $\delta-$distribution of the total energy-momentum

$$P \;\equiv\; p_1 + p_2 +\ \ldots\ + p_n$$

does vindicate the translation invariance of the $n-$point Green's functions in the configuration space.

The disconnected $n-$point Green's functions do involve also trivial parts, that correspond to the absence of any scattering process. Hence, what we are really interested for is the reduction formula for the *connected* Green's functions, that means, the truly interacting part which contribute to the scattering amplitudes. For example, in the 4-point Green's function we find terms which are related to the products of two 2-point Green's functions, *i.e.* two full propagators, and such a term clearly does not describe neither scattering nor interaction. To see this, I first decompose the 4-point Green's function into disconnected and connected parts as shown graphically in Fig. N. 8 The first three graphs represent the unscattered or *straight through* or even *forward* propagation of the particles, albeit with fully interacting or *dressed* propagators, *i.e.* 2-point Green's functions that include all order radiative corrections which describe emission and absorbtion of virtual particles, in accordance with the energy-time uncertainty relation of quantum mechanics.

The final graph represents the processes that give rise to the scattering, once we have again removed the four dressed propagator factors to define an amplitude which is named *truncated* or *amputated* 4-point Green's function.

In conclusion, from the reduction formulæ we have learned that the basic ingradients we have to build up in perturbation theory by means of the Feynman rules, in the aim of computing the scattering cross sections to be compared with the experimental data, are the *connected, truncated, on-shell Green's functions in momentum space.*

3.5 Problems

1. Evaluate the vacuum expectation value

$$\Big\langle\, a_{\,\mathrm{in}}(k_4)\, a_{\,\mathrm{in}}(k_3) : \phi_{\,\mathrm{in}}(x_1)\, \phi_{\,\mathrm{in}}(x_2)\ \ldots\ \phi_{\,\mathrm{in}}(x_{\,n}) : a^{\dagger}_{\,\mathrm{in}}(k_2)\, a^{\dagger}_{\,\mathrm{in}}(k_1)\,\Big\rangle_0$$

in which as usual

$$\phi_{\,\mathrm{in}}(x_\jmath) = \phi^{(-)}_{\,\mathrm{in}}(x_\jmath) + \phi^{(+)}_{\,\mathrm{in}}(x_\jmath) \qquad \jmath = 1, 2, \ldots, n$$

$$\phi^{(-)}_{\,\mathrm{in}}(x_\jmath) = \int D p_\jmath\, a_{\,\mathrm{in}}(p_\jmath)\ \exp\{-\,\mathrm{i}\,p_\jmath\cdot x_\jmath\} \qquad (\text{destruction part})$$

$$\phi^{(+)}_{\,\mathrm{in}}(x_\jmath) = \int D p_\jmath\, a^{\dagger}_{\,\mathrm{in}}(p_\jmath)\ \exp\{+\,\mathrm{i}\,p_\jmath\cdot x_\jmath\} \qquad (\text{creation part})$$

Solution. For $n = 1$ we have

$$\begin{aligned}
&\Big\langle\, a_{\,\mathrm{in}}(k_4)\, a_{\,\mathrm{in}}(k_3)\ : \phi_{\,\mathrm{in}}(x_1) :\ a^{\dagger}_{\,\mathrm{in}}(k_2)\, a^{\dagger}_{\,\mathrm{in}}(k_1)\,\Big\rangle_0 \\
=\ &\Big\langle\, a_{\,\mathrm{in}}(k_4)\, a_{\,\mathrm{in}}(k_3)\, \phi^{(-)}_{\,\mathrm{in}}(x_1)\, a^{\dagger}_{\,\mathrm{in}}(k_2)\, a^{\dagger}_{\,\mathrm{in}}(k_1)\,\Big\rangle_0 \\
+\ &\Big\langle\, a_{\,\mathrm{in}}(k_4)\, a_{\,\mathrm{in}}(k_3)\, \phi^{(+)}_{\,\mathrm{in}}(x_1)\, a^{\dagger}_{\,\mathrm{in}}(k_2)\, a^{\dagger}_{\,\mathrm{in}}(k_1)\,\Big\rangle_0 \\
=\ &\int D p_1\ \exp\{-\,\mathrm{i}\,p_1\cdot x_1\} \\
\times\ &\Big\langle\, a_{\,\mathrm{in}}(k_4)\, a_{\,\mathrm{in}}(k_3)\, a_{\,\mathrm{in}}(p_1)\, a^{\dagger}_{\,\mathrm{in}}(k_2)\, a^{\dagger}_{\,\mathrm{in}}(k_1)\,\Big\rangle_0 \\
+\ &\int D p_1\ \exp\{+\,\mathrm{i}\,p_1\cdot x_1\} \\
\times\ &\Big\langle\, a_{\,\mathrm{in}}(k_4)\, a_{\,\mathrm{in}}(k_3)\, a^{\dagger}_{\,\mathrm{in}}(p_1)\, a^{\dagger}_{\,\mathrm{in}}(k_2)\, a^{\dagger}_{\,\mathrm{in}}(k_1)\,\Big\rangle_0
\end{aligned}$$

Now we get

$$\begin{aligned}
&\Big\langle\, a_{\,\mathrm{in}}(k_4)\, a_{\,\mathrm{in}}(k_3)\, a_{\,\mathrm{in}}(p_1)\, a^{\dagger}_{\,\mathrm{in}}(k_2)\, a^{\dagger}_{\,\mathrm{in}}(k_1)\,\Big\rangle_0 \\
=\ &\Big\langle\, a_{\,\mathrm{in}}(k_4)\, a_{\,\mathrm{in}}(k_3)\, a^{\dagger}_{\,\mathrm{in}}(k_2)\, a_{\,\mathrm{in}}(p_1)\, a^{\dagger}_{\,\mathrm{in}}(k_1)\,\Big\rangle_0 \\
+\ &\Big\langle\, a_{\,\mathrm{in}}(k_4)\, a_{\,\mathrm{in}}(k_3)\, a^{\dagger}_{\,\mathrm{in}}(k_1)\,\Big\rangle_0\ (2\pi)^3\, 2\,\omega(\mathbf{k}_2)\, \delta(\mathbf{p}_1 - \mathbf{k}_2) \\
=\ &\Big\langle\, a_{\,\mathrm{in}}(k_4)\, a_{\,\mathrm{in}}(k_3)\, a^{\dagger}_{\,\mathrm{in}}(k_2)\,\Big\rangle_0\ (2\pi)^3\, 2\,\omega(\mathbf{k}_1)\, \delta(\mathbf{p}_1 - \mathbf{k}_1) \\
+\ &\Big\langle\, a_{\,\mathrm{in}}(k_4)\, a_{\,\mathrm{in}}(k_3)\, a^{\dagger}_{\,\mathrm{in}}(k_1)\,\Big\rangle_0\ (2\pi)^3\, 2\,\omega(\mathbf{k}_2)\, \delta(\mathbf{p}_1 - \mathbf{k}_2)
\end{aligned}$$

It is convenient to introduce the short notation

$$(2\pi)^3\, 2\,\omega\,(\mathbf{k}_2)\,\delta\,(\mathbf{p}_1 - \mathbf{k}_2) = (2\pi)^3\, 2\,\omega\,(\mathbf{p}_1)\,\delta\,(\mathbf{p}_1 - \mathbf{k}_2) = \Delta\,(p_1 - k_2)$$

in such a manner that we can reduce the above vacuum expectation value as follows :

$$\begin{aligned} & \Big\langle\, a_{\,\rm in}\,(k_4)\, a_{\,\rm in}\,(k_3)\, a_{\,\rm in}\,(p_1)\, a^{\dagger}_{\rm in}\,(k_2)\, a^{\dagger}_{\rm in}\,(k_1)\,\Big\rangle_0 \\ = \;\; & \langle\, a_{\,\rm in}\,(k_4)\,\rangle \Big[\,\Delta\,(p_1 - k_2)\,\Delta\,(k_1 - k_3) + \Delta\,(p_1 - k_1)\,\Delta\,(k_2 - k_3)\,\Big] \end{aligned}$$

and quite analogously

$$\begin{aligned} & \Big\langle\, a_{\,\rm in}\,(k_4)\, a_{\,\rm in}\,(k_3)\, a^{\dagger}_{\rm in}\,(p_1)\, a^{\dagger}_{\rm in}\,(k_2)\, a^{\dagger}_{\rm in}\,(k_1)\,\Big\rangle_0 \;=\; \\ & \langle\, a^{\dagger}_{\rm in}\,(k_1)\,\rangle \Big[\,\Delta\,(p_1 - k_3)\,\Delta\,(k_2 - k_4) + \Delta\,(p_1 - k_4)\,\Delta\,(k_2 - k_3)\,\Big] \end{aligned}$$

so that apparently

$$\Big\langle\, a_{\,\rm in}\,(k_4)\, a_{\,\rm in}\,(k_3)\;:\phi_{\,\rm in}\,(x_1):\; a^{\dagger}_{\rm in}\,(k_2)\, a^{\dagger}_{\rm in}\,(k_1)\,\Big\rangle_0 \;\equiv\; 0$$

A straightforward iteration of the above procedure clearly shows that a non-vanishing result solely arises when the number of free fields $\phi_{\,\rm in}\,(x_\jmath)$ exactly matches the total number of the incoming and outgoing wave packets, that is

$$\begin{aligned} & \langle\, a_{\,\rm in}\,(k_4)\, a_{\,\rm in}\,(k_3) : \phi_{\,\rm in}\,(x_1)\,\phi_{\,\rm in}\,(x_2)\;\ldots\;\phi_{\,\rm in}\,(x_n) : a^{\dagger}_{\rm in}\,(k_2)\, a^{\dagger}_{\rm in}\,(k_1)\,\rangle \\ & \langle a_{\,\rm in}(k_4) a_{\,\rm in}(k_3)\phi^{(+)}_{\rm in}(x_1)\phi^{(+)}_{\rm in}(x_2)\,\phi^{(-)}_{\rm in}(x_3)\phi^{(-)}_{\rm in}(x_4) a^{\dagger}_{\rm in}(k_2)\, a^{\dagger}_{\rm in}(k_1)\rangle \\ & \qquad + \;\; \text{permutations}\;\{(\,1 \leftrightarrow 2\,)\,(\,3 \leftrightarrow 4\,)\}\;\begin{cases} = 0 & \text{for } n \neq 4 \\ \neq 0 & \text{for } n = 4 \end{cases} \end{aligned}$$

Chapter 4

Radiative corrections

4.1 Evaluation of Feynman integrals

The Feynman rules lead to loop integrals which are admittedly poorly defined divergent expressions. The divergencies we have to face with are caused by the non−integrable behaviour of the loop integrand functions at very high energy−momentum : these are the *ultra−violet divergencies* of quantum field theories. In the case of field theory models involving massless particles, *e.g.* photons, another kind of low 4−momentum non−integrable singularities indeed appear, the so named *infra−red divergencies*, which will not be treated for the moment in the present context.

The simplest examples arise immediately in the real scalar self-interacting field theory and in the Yukawa spinor-meson field theory. Specifically, from the lowest order expressions (1.22) and (1.23) for the 2−point and 4−point connected Green's functions in momentum space, after truncation of the external free propagators, we can pick out the divergent parts

$$\begin{aligned} \Gamma_2(0) &= (k^2 - m^2)^2\, G_c^{(2)}(k) \qquad \text{(scalar tadpole)} \\ &= \frac{i\lambda}{2} \int \frac{d^4\ell}{(2\pi)^4}\, \frac{i}{\ell^2 - m^2 + \mathrm{i}\varepsilon} \end{aligned} \tag{4.1}$$

$$\begin{aligned} \Gamma_4(k_1,k_2,k_3,k_4) &= \frac{1}{2}(-\mathrm{i}\lambda)^2 \sum_{(\imath\jmath)} (2\pi)^4 \times \qquad (4.2) \\ \int \frac{d^4\ell_1}{(2\pi)^4}\, &\frac{i}{\ell_1^2 - m^2 + \mathrm{i}\varepsilon} \int \frac{d^4\ell_2}{(2\pi)^4}\, \frac{i}{\ell_2^2 - m^2 + \mathrm{i}\varepsilon}\, \delta(\ell_1 + \ell_2 - k_\imath - k_\jmath) \\ &= \prod_{\jmath=1}^{4} (k_\jmath^2 - m^2)\, G_c^{(4)}(k_1,k_2,k_3,k_4) \end{aligned}$$

where the sum $(\imath\jmath)$ runs over the three pairs $(12),(13),(14)$, which turns out to be related, as we shall see further on, to the first radiative correction to the self-interaction coupling λ.

Finally, the first two coefficient of the perturbative expansion for the fermion determinant (1.31) in the Yukawa field theory formally read

$$S_F(0) = \int \frac{dp}{(2\pi)^4}\ \mathrm{tr}\,\frac{i}{\not p - M + \mathrm{i}\varepsilon} \qquad \text{(spinor tadpole)} \tag{4.3}$$

$$\Pi(k) = i\,g^2 \int \frac{dp}{(2\pi)^4}\ \mathrm{tr}\,\frac{i}{\not p - M + \mathrm{i}\varepsilon}\cdot\frac{i}{\not p + \not k - M + \mathrm{i}\varepsilon} \tag{4.4}$$

which corresponds to the so named $\pi-$meson (pion) self$-$energy.

In order to give a precise mathematical meaning to the above listed ill$-$defined integral expressions, we have to introduce from the outset some kind of *regularization procedure*, just to the aim of building up absolutely convergent loop integrals. Thus, I will briefly review here below the most commonly used ultraviolet regulators, by applying the latter ones to the above written paradigmatic simple divergent loop integrals.

4.1.1 Cut$-$off regularization

This is the most intuitive and physically motivated ultraviolet regulator, that I have already employed in the discussion of the vacuum energy and the cosmological constant – see the First Semester Course. It is based upon the rather plausible belief that the validity of the principles of modern quantum field theory, as well as the classical theory of gravitation based upon Einstein general relativity, can not be pushed beyond a very high energy scale such as the Planck scale

$$\begin{aligned} M_P &= \sqrt{\hbar c/G_N} = 1.220\,90(9)\times 10^{19}\ \mathrm{GeV/c^2} \\ &= 2.176\,45(16)\times 10^{-11}\ \mathrm{g} \\ G_N &= 6.674\,2(10)\times 10^{-8}\ \mathrm{cm^3\ g^{-1}\ s^{-2}} \end{aligned}$$

where G_N is the newtonian gravitational constant. The matter is that at the Plack scale some new physics is expected to govern the quantum gravity phenomena, a realm which does not seem to be experimentally accessible nowadays [1]. If we trust in general relativity and in quantum field theory

[1] Nonetheless, at the moment there are indeed big efforts in trying to detect ultra-high-energy cosmic rays (UHECR), ultra-high-energy $\gamma-$ray bursts as well as black-hole particle physics effects at LHC, due to the possible existence of space-time extra-dimensions.

up to the Planck scale but not beyond, it turns out to be quite natural to cut−off the loop integrations at a very high wave number of the order $K \simeq (8\pi G_N)^{-1/2}$.

Consider therefore the cut−off scalar tadpole

$$
\begin{aligned}
\Gamma_2(0) &= (k^2 - m^2)^2 G_c^{(2)}(k) \\
&= \frac{i\lambda}{2}\int \frac{d\boldsymbol{\ell}}{(2\pi)^4}\,\theta(K^2 - \boldsymbol{\ell}^2)\int_{-\infty}^{\infty} d\ell_0\,\frac{i}{\ell_0^2 - \boldsymbol{\ell}^2 - m^2 + \mathrm{i}\varepsilon} \\
&= \frac{i\lambda}{4}\int \frac{d\boldsymbol{\ell}}{(2\pi)^3}\,\theta(K^2 - \boldsymbol{\ell}^2)\left(\boldsymbol{\ell}^2 + m^2\right)^{-1/2} \\
&= \frac{i\lambda}{8\pi^2}\int_0^K d\ell\,\ell^2\left(\ell^2 + m^2\right)^{-1/2} \\
&= i\lambda\frac{d}{dm^2}\int_0^K \frac{d\ell}{4\pi^2}\,\ell^2\sqrt{\ell^2 + m^2} \;=\; i\lambda\,\frac{d\langle\,\rho\,\rangle}{dm^2} \qquad (4.5)
\end{aligned}
$$

where the quantity $\langle\,\rho\,\rangle$ is nothing but the vacuum energy density that I have already introduced in the first part of these notes and which is related to the so called zero−point energy of the quantized scalar field. From [21] eq. **2.272**3. p. 105 we obtain

$$
\begin{aligned}
\mathtt{reg}\,\Gamma_2(0;K) &= \frac{i\lambda K^2}{(4\pi)^2}\left\{\sqrt{1 + m^2/K^2} - \frac{m^2}{K^2}\right. \\
&\times \left.\left[\ln\frac{K}{m} + \ln\left(1 + \sqrt{1 + m^2/K^2}\right)\right]\right\} \qquad (4.6)
\end{aligned}
$$

and from the explicit expression of the vacuum energy density

$$
\langle\,\rho\,\rangle = \frac{K^4}{16\pi^2} + \frac{K^2 m^2}{16\pi^2} - \frac{m^4}{32\pi^2}\left[\ln\frac{K}{m} - \frac{1}{4} + \ln 2 + O\left(\frac{m}{K}\right)^2\right]
$$

we eventually understand the ultraviolet cut−off regularized scalar tadpole as follows : namely,

$$
\mathtt{reg}\,\Gamma_2(0;K) \stackrel{\mathrm{def}}{=} \frac{i\lambda}{16\pi^2}\left\{K^2 - m^2\left[\ln\frac{K}{m} - \frac{1}{2} + \ln 2 + O\left(\frac{m}{K}\right)^2\right]\right\} \qquad (4.7)
$$

which means a quadratic divergence and a logarithmic divergence at the Planck scale. It follows that if we remove the zero−point vacuum energy through normal ordering then the divergent scalar tadpole disappears.

The spinor tadpole can be treated in the very same way: namely,

$$
\mathtt{reg}\,S^F(0;K) = \frac{\mathrm{i}}{(2\pi)^4}\int d\mathbf{p}\;\theta(K^2 - \mathbf{p}^2)\int_{-\infty}^{\infty}\frac{\mathrm{tr}\,(\not{p} + M)\,dp_0}{p_0^2 - \mathbf{p}^2 - M^2 + \mathrm{i}\varepsilon}
$$

$$= \frac{M}{2\pi^2}\left\{K^2 - M^2\left[\ln\frac{K}{M} - \frac{1}{2} + \ln 2 + O\left(\frac{M}{K}\right)^2\right]\right\} \tag{4.8}$$

which means, as expected, that we have again a quadratic divergence and a logarithmic divergence at the Planck scale.

4.1.2 Pauli−Villars regularization

This method for the ultraviolet regularization of fermion cycles has been introduced in the quantum field theory by one of the main father−founders and one of his students

Wolfgang Pauli & Felix Villars
(Swiss Federal Institute of Technology, Zurich, Switzerland)
On the Invariant Regularization in Relativistic Quantum Theory
Review of Modern Physics **21**, 434 - 444 (1949) [Issue 3 − July 1949]

See also : Claude Itzykson and Jean−Bernard Zuber, *Quantum Field Theory*, McGraw-Hill, New York, 1980, § **7-1-1** p. 319 ; Nicolai Nicolaievic Bogoliubov and D.M. Shirkov, *Introduction to the theory of quantized fields*, Interscience Publishers, New York, 1959, § 30.2, p. 364 ; Ludwig Dimitrievich Fadde'ev & Andrei Alexe'evich Slavnov, *Gauge fields. Introduction to quantum theory*, Benjamin, Reading (MA), 1980, § **4.3**, p. 131.

To implement it consider the spinor propagator in momentum space

$$S_F(p\,,\,M) = \frac{\mathrm{i}}{\not p - M + \mathrm{i}\varepsilon} = \frac{i\,(\not p + M)}{p^2 - M^2 + \mathrm{i}\varepsilon}$$

and let me define the *pion self−energy*, or the vacuum polarization scalar, for the Yukawa theory with the Pauli−Villars regularisation, *viz.*

$$\mathtt{reg}\,\Sigma(k\,;\Lambda) \stackrel{\mathrm{def}}{=} \mathrm{i}g^2\int\frac{\mathrm{d}^4p}{(2\pi)^4}\sum_{s=0}^{S} C_s\,\mathrm{tr}\,S_F(p\,,\,M_s)\,S_F(p+k\,,\,M_s)$$

where $M_0 = M$, $C_0 = 1$ while $\{\,M_s \equiv \lambda_s M \mid \lambda_s \gg 1\ (\,s = 1,2,\ldots,S\,)\}$ is a collection of very large auxiliary masses. The set of constants C_s $(\,s = 1,2,\ldots,S\,)$ will be suitably selected in such a manner to eventually remove the ultraviolet divergencies, as we shall see in the sequel. Further, we have denoted the collection of very large auxiliary masses M_s $(\,s = 1,2,\ldots,S\,)$ by the symbol Λ . Then we get

$$\mathtt{reg}\,\Sigma(k\,;\Lambda) =$$
$$-\ i\,g^2\int\frac{\mathrm{d}^4p}{(2\pi)^4}\sum_{s=0}^{S}\frac{C_s\,\mathrm{tr}\,[\,(\not p + M_s)\,(\not p + \not k + M_s)\,]}{(p^2 - M_s^2 + \mathrm{i}\varepsilon)\,[\,(p+k)^2 - M_s^2 + \mathrm{i}\varepsilon\,]}$$

Taking into account that we have

$$\mathrm{tr}\,(\gamma^{\mu}\gamma^{\nu}) \;=\; g^{\mu\nu}\,\mathrm{tr}\,\mathbb{I} \;=\; 4\,g^{\mu\nu} \qquad\qquad \mathrm{tr}\,\gamma^{\lambda} \;=\; 0$$

we can write

$$\begin{aligned}&\mathtt{reg}\,\Sigma(k\,;\Lambda) \;= \\ -\;\; & 4i\,g^2\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\;\sum_{s=0}^{S}\;\frac{C_s\,(p^2+p\cdot k+M_s^2\,)}{(p^2-M_s^2+\mathrm{i}\varepsilon\,)\,[\,(p+k\,)^2-M_s^2+\mathrm{i}\varepsilon\,]}\end{aligned}$$

It is convenient to introduce the *Feynman parametrization formula*

$$\begin{aligned}&\frac{1}{(p^2-M_s^2+\mathrm{i}\varepsilon\,)\,[\,(p+k)^2-M_s^2+\mathrm{i}\varepsilon\,]} \;= \\ &\int_0^1\frac{\mathrm{d}x}{\{\,x\,(p^2-M_s^2\,)+(1-x)\,[\,(p+k)^2-M_s^2\,]+\mathrm{i}\varepsilon\,\}^2} \\ &=\;\int_0^1\frac{\mathrm{d}x}{[\,p^2-M_s^2+2p\cdot k\,(1-x)+(1-x)\,k^2\,]^2}\end{aligned}$$

and by exchanging the integrals

$$\begin{aligned}\mathtt{reg}\,\Sigma(\,k\,;\,\Lambda\,) \;=\; -\,4i\,g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4 p}{(2\pi)^4} \\ \sum_{s=0}^{S}\;\frac{C_s\,(p^2+p\cdot k+M_s^2\,)}{[\,p^2-M_s^2+2p\cdot k\,(1-x)+(1-x)\,k^2\,]^2}\end{aligned}$$

Let us shift the integration variable

$$p\;\mapsto\;p'=p+k\,(1-x)$$

which yields

$$\begin{aligned}\mathtt{reg}\,\Sigma(\,k\,;\,\Lambda\,) = -\,4i\,g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4 p'}{(2\pi)^4} \\ \sum_{s=0}^{S}\;C_s\;\frac{M_s^2-x\,(1-x)\,k^2+p'^2-p'\cdot k\,(1-2x)}{[\,p'^2-M_s^2+x\,(1-x)\,k^2\,]^2}\end{aligned}$$

The very last term in the numerator is odd and vanishes after integration. Thus we are left with

$$\mathtt{reg}\,\Sigma(k;\Lambda) = -\,4ig^2\int_0^1\mathrm{d}x\sum_{s=0}^{S}\,C_s\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\,\frac{p^2+M_s^2-x\,(1-x)\,k^2}{[\,p^2-M_s^2+x\,(1-x)\,k^2\,]^2}$$

Notice that for $k^2 < 0$ the integrand has two real double poles at

$$p_0 = \pm\sqrt{\mathbf{p}^2 + k^2\, R(x, a_s)}$$

$$R(x, a_s) = x^2 - x + a_s \qquad\qquad a_s = \frac{M_s^2}{k^2} < 0$$

To go further on let me first wisely perform the Wick rotation, that is, let us consider the closed oriented contour γ^+ in the complex energy−plane of fig. N 12. Notice that, thanks to the causal prescription, the two real double poles of the integrand lie outside the contour γ^+ for $k^2 < 0$. Since the contributions due to the two arcs of the large circle of radius R do vanish when $R \to \infty$ we obtain

$$\mathtt{reg}\,\Sigma_E(k_E\,;\Lambda) = 4g^2 \int \frac{\mathrm{d}^4 p_E}{(2\pi)^4} \sum_{s=0}^{S} C_s\, \frac{M_s^2 + x(1-x)\,k_E^2 - p_E^2}{[\,p_E^2 + M_s^2 + x(1-x)\,k_E^2\,]^2}$$

in which I have set $k^0 = i\,k_4\,,\; p^0 = i\,p_4$ and $p_{E\mu} = (\,\mathbf{p}\,,\,p_4\,)\,,\; p_E^2 = \mathbf{p}^2 + p_4^2\,.$

Let us now consider the Pauli−Villars regularization for the generating integral representation that reads

$$I_n\,(z_E\,,\,\Lambda) \;\equiv\; (-1)^n \int \frac{\mathrm{d}^4 p_E}{(2\pi)^4} \sum_{s=0}^{S} C_s\, \frac{\exp\{\,i\,p_E\cdot z_E\,\}}{(\,p_E^2 + \Delta_s^2\,)^{\,n}} \tag{4.9}$$

in which I have set for short $\Delta_s^2 \;=\; M_s^2 + x\,(1-x)\,k_E^2\,.$ It follows that we can write

$$\mathtt{reg}\,\Sigma_E(k_E\,;\Lambda) \;=\; 4g^2 \int_0^1 \mathrm{d}x \lim_{z_E\to 0} \big(\,2\Delta^2\,I_2 + I_1\big)\,\exp\{\,i\,k_E\cdot z_E\,\}$$

where

$$\Delta^2\,I_2 \equiv \int \frac{\mathrm{d}^4 p_E}{(2\pi)^4} \sum_{s=0}^{S} C_s\,\Delta_s^2\, \frac{\exp\{\,i\,p_E\cdot z_E\,\}}{(\,p_E^2 + \Delta_s^2\,)^{\,2}}$$

Taking the Mellin transform we find

$$\begin{aligned} I_n\,(z_E\,,\,\Lambda) \;&=\; \frac{1}{(2\pi)^4} \int \mathrm{d}^4 p_E \sum_{s=0}^{S} C_s\, \frac{(-1)^n}{\Gamma(n)} \\ &\times\; \int_0^\infty \mathrm{d}t\; t^{\,n-1} \exp\{-t\,p_E^2 - t\,\Delta_s^2 + i\,p_E\cdot z_E\} \\ &=\; \frac{(-1)^n}{\Gamma(n)} \int_0^\infty \mathrm{d}t\; t^{\,n-1} \sum_{s=0}^{S} C_s\, \exp\{-\,t\,\Delta_s^2\} \end{aligned}$$

$$\times \quad \frac{1}{(2\pi)^4}\int \mathrm{d}^4 p_E \;\exp\left\{ -t\left(p_E - i\,\frac{z_E}{2t}\right)^2 - \frac{z_E^2}{4t}\right\}$$
$$= \frac{(-1)^n}{16\pi^2\,\Gamma(n)}\int_0^\infty \mathrm{d}t\; t^{n-3}\sum_{s=0}^{S} C_s\;\exp\{-t\,\Delta_s^2 - z_E^2/4t\}$$
$$= \frac{2(-1)^n}{16\pi^2\,\Gamma(n)}\sum_{s=0}^{S} C_s\;\left(\frac{2\,\Delta_s}{|\,z_E\,|}\right)^{2-n} K_{2-n}(\Delta_s\,|z_E\,|)$$

where

$$|\,z_E\,| = \sqrt{\mathbf{z}^2 + z_4^2} = \sqrt{\mathbf{z}^2 - z_0^2} = \sqrt{-\,z^2} \qquad (z^2 < 0)$$

For $n = 1$ and $z_s \equiv \Delta_s\,|\,z_E\,|$ we obtain

$$I_1\,(z_E\,,\,\Lambda) = -\left(\frac{1}{2\pi z_E}\right)^2 \sum_{s=0}^{S} C_s\; z_s\,K_1\,(z_s)$$
$$= \left(\frac{1}{2\pi z_E}\right)^2 \sum_{s=0}^{S} C_s\; z_s\,\frac{d}{dz_s}\,K_0\,(z_s)$$
$$I_2\,(z_E\,,\,\Lambda) = \frac{1}{8\pi^2}\sum_{s=0}^{S} C_s\,K_0\,(z_s) \tag{4.10}$$

where K_0 is the modified Bessel function of the third kind, also named Basset−McDonald function, of order zero, the series representation of which is provided by

$$K_0(z) = \sum_{k=0}^{\infty}\;\frac{1}{(k\,!)^2}\left(\frac{z}{2}\right)^{2k}\left[\psi\,(k+1) - \ln\frac{z}{2}\right]$$

while

$$K_1\,(z) = \frac{1}{z} + \sum_{k=0}^{\infty}\;\left(\frac{z}{2}\right)^{2k+1}\frac{1}{k!(k+1)!}\left[\ln\frac{z}{2} - \frac{1}{2}\,\psi\,(k+1) - \frac{1}{2}\,\psi\,(k+2)\right]$$

Now if we choose the auxiliary constants so that

$$\sum_{s=0}^{S} C_s = 0 \qquad\qquad \sum_{s=0}^{S} C_s\,\lambda_s^2 = 0 \tag{4.11}$$

then we find for $z_E \to 0$

$$I_1 + 2\Delta^2\,I_2 = \left(\frac{1}{2\pi z_E}\right)^2 \sum_{s=0}^{S} C_s\,\left[\,z_s^2\,K_0\,(z_s) \;-\; z_s\,K_1\,(z_s)\,\right]$$

$$
\begin{aligned}
&= \left(\frac{1}{2\pi z_E}\right)^2 \sum_{s=0}^{S} C_s\, z_s \left(z_s + \frac{d}{dz_s}\right) K_0(z_s) \\
&= -\,\frac{3}{4}\left(\frac{1}{2\pi}\right)^2 \sum_{s=0}^{S} C_s\, \Delta_s^2 \ln z_s \;+\; O(z_E^2)
\end{aligned}
$$

and finally

$$
\begin{aligned}
&\lim_{z_E \to 0} \{ 2\Delta^2 I_2(z_E\,, \Lambda) + I_1(z_E\,, \Lambda) \} \;= \\
&-\; \sum_{s=0}^{S} \frac{3C_s}{16\pi^2} \left[M^2\lambda_s^2 + x(1-x)\, k_E^2 \right] \ln \left[\lambda_s^2 + x(1-x)\, k_E^2 / M^2 \right] \\
&\sum_{s=1}^{S} C_s \;=\; -1 \;=\; \sum_{s=1}^{S} C_s\, \lambda_s^2 \qquad\qquad C_0 = 1 = \lambda_0
\end{aligned}
\tag{4.12}
$$

Turning back to the invariant polarization function we obtain

$$
\begin{aligned}
\texttt{reg}\,\Sigma_E(k_E\,; \Lambda) \;=\; &-\; \frac{3g^2}{4\pi^2}\int_0^1 \mathrm{d}x \sum_{s=0}^{S} C_s \left[M^2\lambda_s^2 + x(1-x)\, k_E^2 \right] \\
&\times\; \left\{ \ln \lambda_s^2 + \ln \left[1 + x(1-x)\, k_E^2 / M_s^2 \right] \right\}
\end{aligned}
$$

and thereby

$$
\begin{aligned}
\texttt{reg}\,\Sigma_E(k_E\,; \Lambda) \;=\; &-\; \frac{g^2}{8\pi^2} \sum_{s=1}^{S} C_s \ln \lambda_s^2 \left(k_E^2 + 6M_s^2 \right) \\
&+\; \frac{3g^2}{4\pi^2}\int_0^1 \mathrm{d}x \;\left[M^2 + x(1-x)\, k_E^2 \right] \\
&\times\; \ln \left[1 + x(1-x)\, \frac{k_E^2}{M^2} \right] \\
&+\; \frac{3g^2}{4\pi^2}\int_0^1 \mathrm{d}x \; \sum_{s=1}^{S} C_s \left[M_s^2 + x(1-x)\, k_E^2 \right] \\
&\times\; \ln \left[1 + x(1-x)\, \frac{k_E^2}{M_s^2} \right]
\end{aligned}
$$

which clearly suggests how to segregate the divergent and finite parts in the limit $\lambda_s \to \infty$ ($s = 1, 2, \ldots, S$) of very large unphysical auxiliary fermion masses. As a matter of fact, we can rewrite the very last term of the right−hand−side of the above expression in the form

$$\frac{3g^2}{4\pi^2}\int_0^1 dx\ x(1-x)\,k_E^2\sum_{s=1}^{S}C_s\ +\ O(\lambda_s^{-2})\ =\ -\frac{g^2\,k_E^2}{8\pi^2}+O(\lambda_s^{-2})$$

Then we eventually come to the suggestive result

$$\begin{aligned}\texttt{reg}\,\Sigma_E(k_E\,;\Lambda)\ &=\ \frac{g^2}{8\pi^2}\left[k_E^2-\sum_{s=1}^{S}\ C_s\ \ln\lambda_s^2\left(k_E^2+6M_s^2\right)\right]\\ -\ \frac{3g^2}{4\pi^2}\int_0^1 dx\ &\left[M^2+x(1-x)\,k_E^2\right]\ \ln\left[1+x(1-x)\,\frac{k_E^2}{M^2}\right]\end{aligned}$$

and going back to the Minkowski space−time

$$\begin{aligned}\texttt{reg}\,\Sigma(k\,;\Lambda)\ &=\ \frac{g^2}{8\pi^2}\sum_{s=1}^{S}\ C_s\ \ln\lambda_s^2\left(k^2-6M_s^2\right)\\ -\ &\frac{g^2}{8\pi^2}\left[k^2+(6M^2-k^2)\ \ln\frac{k^2}{M^2}\right]\\ -\ &\frac{3g^2}{4\pi^2}\int_0^1 dx\ \left[M^2-x(1-x)\,k^2\right]\ \ln(x^2-x+M^2/k^2)\end{aligned}\tag{4.13}$$

It is convenient to rewrite the above expression in a more convenient and standard form. To this purpose let me define the quantities

$$a\ \equiv\ \frac{M^2}{k^2}\qquad\qquad R\ \equiv\ a-x+x^2\tag{4.14}$$

$$\bar{I}_0\ =\ \int_0^1 \mathrm{d}x\ \ln\frac{a}{R}\qquad\qquad \bar{I}_2\ =\ \int_0^1 \mathrm{d}x\,x(1-x)\ \ln\frac{a}{R}\tag{4.15}$$

so that the vacuum polarization function for the Yukawa field theory in the Pauli−Villars regularisation can be eventually cast in the simple form

$$\begin{aligned}\texttt{reg}\,\Sigma(k,M\,;\Lambda)\ &=\ \frac{g^2}{8\pi^2}\sum_{s=1}^{S}\ C_s\ \ln\lambda_s^2\left(k^2-6M_s^2\right)\\ &-\ \frac{g^2k^2}{8\pi^2}\ -\ \frac{3g^2}{4\pi^2}\,(M^2\,\bar{I}_0-k^2\,\bar{I}_2)\end{aligned}\tag{4.16}$$

Consider the finite part of the pion self−energy

$$\texttt{reg}\,\widehat{\Sigma}\,(k^2;M)\ =\ -\frac{g^2k^2}{8\pi^2}\ -\ \frac{3g^2}{4\pi^2}\int_0^1 \mathrm{d}x\,[M^2-k^2\,x(1-x)]\ \ln\frac{k^2R}{M^2}$$

and let me look at the analytic structure of the integrand in the complex $s-$plane with $\Re\mathrm{e}\, s = k^2$. One can readily verify that the argument of the logarithm is always positive definite for $k^2 < 4M^2$ and $0 \le x \le 1$, while it becomes negative definite for

$$\Re\mathrm{e}\, s > 4M^2 \ \vee\ \frac{1}{2} - \frac{\beta}{2} < x < \frac{1}{2} + \frac{\beta}{2} \qquad \beta = \sqrt{1 - \frac{4M^2}{k^2}} \qquad 0 < \beta < 1$$

It follows that for $k^2 < 4M^2$ the vacuum polarization invariant function is real and analytic $\forall\, x \in [\,0, 1\,]$, while the logarithm develops a branch point when $R = 0$, viz.,

$$\{k^2 = 4M^2 \ \vee\ x = \tfrac{1}{2}\} \cup \{k^2 > 4M^2 \ \vee\ x = \tfrac{1}{2} \pm \tfrac{\beta}{2}\}$$

Notice that $k^2 = 4M^2$ precisely corresponds to the threshold for a creation of a fermion$-$antifermion real pair, in such a manner that the complex $s-$plane has a cut just above the threshold, *i.e.* for $\Re\mathrm{e}\, s > 4M^2$. As a consequence, the imaginary part of the the vacuum polarization invariant function can be readily obtained above/below the cut by

$$\ln\left[\frac{k^2 \pm \mathrm{i}0}{M^2}\left(x^2 - x + \frac{M^2}{k^2 \pm \mathrm{i}0}\right)\right] = \ln\left[\frac{k^2}{M^2}(-R)\right] \mp i\pi \qquad (R < 0)$$

which yields

$$\begin{aligned} \Im\mathrm{m}\, \widehat{\Sigma}(k^2 \pm \mathrm{i}0, M^2) &= \pm\frac{3g^2}{4\pi}\int_{(1-\beta)/2}^{(1+\beta)/2} \mathrm{d}x\ [M^2 - k^2 x(1-x)] \\ &= \pm\frac{g^2k^2}{8\pi}\left(1 - \frac{4M^2}{k^2}\right)^{3/2} \end{aligned} \tag{4.17}$$

Moreover, the parametric integrals of the Appendix eventually yield

$$\mathtt{reg}\,\widehat{\Sigma}(k^2; M) = M^2\,\frac{3g^2}{4\pi^2}\,\bar{I}_0 \ -\ k^2\,\frac{3g^2}{4\pi^2}\,\left(\tfrac{1}{6} + \bar{I}_2\right) \tag{4.18}$$

The first addendum in the right-hand-side of the above equality is closely related to the invariant polarization function of Quantum Electrodynamics and can be rewritten for $0 < k^2 < 4M^2$ as

$$\frac{g^2k^2}{4\pi^2}\left\{-\frac{1}{3} + \left(1 + \frac{2M^2}{k^2}\right) \times\left[\left(\frac{4M^2}{k^2} - 1\right)^{1/2} \mathrm{arcctg}\left(\frac{4M^2}{k^2} - 1\right)^{1/2} - 1\right]\right\}$$

Exercise : let me compare the second and third line of the above equalities with eq. (7-9) § **7-1-1** p. 323 of the book by Claude Itzykson and Jean−Bernard Zuber, *Quantum Field Theory*, McGraw−Hill, New York, 1980. We find

$$
\begin{aligned}
-\,k^2\,\frac{3g^2}{4\pi^2}\,\bar I_2 &= \frac{g^2k^2}{2\cdot 4\pi^2}\int_0^1\frac{dx}{R}\;\left(4x^4-8x^3+3x^2\right)\\
&= \frac{g^2}{4\pi}\cdot\frac{k^2}{2\pi}\left\{\frac{44}{6}-4a+\frac{4-16a+8a^2}{2}\int_0^1\frac{dx}{R}\right.\\
&\left.-12-(4-12a)\int_0^1\frac{dx}{R}+3+\frac{3-6a}{2}\int_0^1\frac{dx}{R}\right\}\\
= \frac{g^2}{4\pi}\cdot\frac{k^2}{2\pi}&\left\{-\,\frac{5}{3}-\frac{4M^2}{k^2}+\left[\frac{M^2}{k^2}\left(1+\frac{4M^2}{k^2}\right)-\frac{1}{2}\right]\int_0^1\frac{dx}{R}\right\}
\end{aligned}
$$

for $0<k^2<4M^2$ we have

$$
\int_0^1\frac{dx}{R}\;=\;\frac{4}{\sqrt\Delta}\,\mathrm{arcctg}\sqrt\Delta \qquad\qquad \Delta=\frac{4M^2}{k^2}-1
$$

and thereby

$$
\begin{aligned}
-\,k^2\,\frac{3g^2}{4\pi^2}\,\bar I_2 &= \frac{g^2k^2}{2\cdot 4\pi^2}\int_0^1\frac{dx}{R}\;\left(4x^4-8x^3+3x^2\right)\;=\\
\frac{g^2}{4\pi}\cdot\frac{k^2}{2\pi}&\left\{-\,\frac{5}{3}-\frac{4M^2}{k^2}+\left[\frac{4M^2}{k^2}\,(2+\Delta)-2\right]\,\frac{1}{\sqrt\Delta}\,\mathrm{arcctg}\sqrt\Delta\right\}\\
&= \frac{g^2}{4\pi}\cdot\frac{k^2}{2\pi}\left\{-\,\frac{5}{3}-\frac{4M^2}{k^2}+2\left(1+\frac{2M^2}{k^2}\right)\,\sqrt\Delta\,\mathrm{arcctg}\sqrt\Delta\right\}\\
&= \frac{g^2}{4\pi}\cdot\frac{k^2}{2\pi}\left\{\frac{1}{3}+2\left(1+\frac{2M^2}{k^2}\right)\left[\left(\frac{4M^2}{k^2}-1\right)^{1/2}\right.\right.\\
&\left.\left.\times\;\mathrm{arcctg}\left(\frac{4M^2}{k^2}-1\right)^{1/2}-1\right]\right\}
\end{aligned}
$$

which is in perfect agreement. *Quod Erat Demonstrandum*

Moreover, the first addendum in the right hand side of eq. (4.18) can be recast in the form

$$
M^2\,\frac{3g^2}{4\pi^2}\,\bar I_0=\frac{3g^2M^2}{2\pi^2}\left\{1-\left(\frac{4M^2}{k^2}-1\right)^{1/2}\mathrm{arcctg}\left(\frac{4M^2}{k^2}-1\right)^{1/2}\right\}
$$

To sum up, putting all together we eventually obtain

$$
\begin{aligned}
\mathtt{reg}\,\Sigma(k^2,M^2;\Lambda) &= \frac{g^2}{8\pi^2}\sum_{s=1}^{S}C_s\,\ln\lambda_s^2\left(k^2-6M_s^2\right) \qquad (4.19)\\
+\;\;\frac{3g^2M^2}{2\pi^2}&\left\{1-\left(\frac{4M^2}{k^2}-1\right)^{1/2}\mathrm{arcctg}\left(\frac{4M^2}{k^2}-1\right)^{1/2}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{g^2k^2}{4\pi^2}\left\{\frac{1}{3}-\left(1+\frac{2M^2}{k^2}\right)\left[\left(\frac{4M^2}{k^2}-1\right)^{1/2}\text{arcctg}\left(\frac{4M^2}{k^2}-1\right)^{1/2}-1\right]\right\} \\
&= \frac{3g^2}{4\pi^2}\sum_{s=1}^{S} C_s \ln \lambda_s^2\left(\tfrac{1}{6}k^2-M_s^2\right)+\frac{g^2}{\pi^2}\left(M^2-\tfrac{1}{3}k^2\right) \\
&- \frac{g^2k^2}{4\pi^2}\left(\frac{4M^2}{k^2}-1\right)^{3/2}\text{arcctg}\left(\frac{4M^2}{k^2}-1\right)^{1/2} \qquad (4.20)
\end{aligned}
$$

4.1.3 Dimensional regularization

The technique of dimensional regularization has been invented by

Gerardus 't Hooft and Martinus Justinus Godefriedus Veltman
Regularization and renormalization of gauge fields
Nuclear Physics **44B**, 189 (1972)
C.G. Bollini and J.J. Giambiagi, Physics Letters **40B**, 566 (1972)
G. M. Cicuta and E. Montaldi, Lettere Nuovo Cimento **4**, 329 (1972)
J. F. Ashmore, Lettere Nuovo Cimento **4**, 289 (1972)

The basic idea behind this tool is very simple : by lowering the number of dimensions over which one integrates, it might happen that the divergences trivially disappear. Then we can give a precise meaning to some divergent loop integral trough the method of analytic continuation in the number of space−time dimensions D that could be eventually turned into a complex number $2\omega \in \mathbb{C}$. In so doing, the divergences appear as poles in the complex $\omega-$plane. Let me do a simple calculation to understand how this technique is at work.

Consider a $D-$dimensional space−time with $D-1$ spatial dimensions and one time dimension. Then we can perform the Wick rotation to calculate any Feynman integral and produce an absolutely convergent integral in a $2\omega-$dimensional euclidean space with ω sufficiently small. A typical example is provided by

$$
I = \mu^{4-2\omega}\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\ (k_E^2+\Delta)^{-2}
$$

with $\Delta > 0$, which is absolutely convergent for $\omega < 2$. The arbitrary mass scale μ has been introduced with a suitable power, in such manner to deal with a dimensionless quantity I. The spherical polar coordinates of $k_{E\mu}$ are

$k, \phi, \theta_1, \theta_2, \ldots, \theta_{2\omega-2}$ and we have

$$\begin{cases} k_1 = k\cos\theta_1 \\ k_2 = k\sin\theta_1\cos\theta_2 \\ k_3 = k\sin\theta_1\sin\theta_2\cos\theta_3 \\ \cdots\cdots\cdots \\ k_{2\omega-1} = k\sin\theta_1\sin\theta_2\cdots\sin\theta_{2\omega-2}\cos\phi \\ k_{2\omega} = k\sin\theta_1\sin\theta_2\cdots\sin\theta_{2\omega-2}\sin\phi \end{cases}$$

with $0 \le \theta_\imath \le \pi$ for $\imath = 1, 2, \ldots, 2\omega - 2$ and $0 \le \phi \le 2\pi$ while $k = |k_E| \ge 0$. It turns out that

$$\frac{\partial(k_1, k_2, \cdots, k_{2\omega})}{\partial(k, \phi, \theta_1, \cdots, \theta_{2\omega-2})} = k^{2\omega-1}(\sin\theta_1)^{2\omega-2}(\sin\theta_2)^{2\omega-3}\cdots(\sin\theta_{2\omega-2})$$

Hence we immediately obtain

$$\begin{aligned} I &= \frac{\mu^{4-2\omega}}{(2\pi)^{2\omega}}\int_0^\infty \frac{dk\ k^{2\omega-1}}{(k^2+\Delta)^2} \\ &\times\ (2\pi)\prod_{\jmath=1}^{2\omega-2}\int_0^\pi d\theta_\jmath\ (\sin\theta_\jmath)^{2\omega-\jmath-1} \end{aligned}$$

Now we have

$$\begin{aligned} \int_0^\pi d\theta_\jmath\ (\sin\theta_\jmath)^{2\omega-\jmath-1} &= 2\int_0^1 dt_\jmath\ (1-t_\jmath^2)^{\omega-1-\jmath/2} \\ &= \int_0^1 dy\ y^{-1/2}(1-y)^{\omega-1-\jmath/2} \\ &= B(1/2, \omega-\jmath/2) \\ &= \sqrt{\pi}\ \frac{\Gamma(\omega-j/2)}{\Gamma(\omega-j/2+1/2)} \end{aligned}$$

where

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

is the Euler Beta function, so that

$$\begin{aligned} &(2\pi)\prod_{\jmath=1}^{2\omega-2}\int_0^\pi d\theta_\jmath\ (\sin\theta_\jmath)^{2\omega-\jmath-1} \\ &= \frac{2\pi^\omega\,\Gamma(1)\Gamma(3/2)\Gamma(2)\cdots\Gamma(\omega-1/2)}{\Gamma(3/2)\Gamma(2)\cdots\Gamma(\omega-1/2)\Gamma(\omega)} = \frac{2\pi^\omega}{\Gamma(\omega)} \end{aligned} \qquad (4.21)$$

and thereby

$$
\begin{aligned}
I &= \frac{2\pi^{\omega}\,\mu^{4-2\omega}}{\Gamma(\omega)\,(2\pi)^{2\omega}}\int_0^\infty \frac{k\,dk\;(k^2)^{\omega-1}}{(k^2+\Delta)^2} \\
&= \frac{\mu^{4-2\omega}}{(4\pi)^{\omega}\,\Gamma(\omega)}\int_0^\infty \frac{dq\;q^{\omega-1}}{(q+\Delta)^2} \\
&= \frac{\mu^{4-2\omega}}{(4\pi)^{\omega}\,\Gamma(\omega)}\left(-\frac{d}{d\Delta}\right)\int_0^\infty \frac{dq\;q^{\omega-1}}{q+\Delta} \\
&= \frac{\mu^{4-2\omega}}{(4\pi)^{\omega}\,\Gamma(\omega)}\left(-\frac{d}{d\Delta}\right)\int_0^\infty dq\;q^{\omega-1}\int_0^\infty dt\;\;\exp\{-tq-t\Delta\} \\
&= \frac{\mu^{4-2\omega}}{(4\pi)^{\omega}\,\Gamma(\omega)}\left(-\frac{d}{d\Delta}\right)\int_0^\infty dt\;\mathrm{e}^{-t\Delta}\int_0^\infty dq\;q^{\omega-1}\,\mathrm{e}^{-tq} \\
&= \frac{\mu^{4-2\omega}}{(4\pi)^{\omega}}\int_0^\infty dt\;t^{1-\omega}\,\mathrm{e}^{-t\Delta} = \frac{\Gamma(2-\omega)}{(4\pi)^{\omega}}\left(\frac{\mu^2}{\Delta}\right)^{2-\omega} \qquad (4.22)
\end{aligned}
$$

which is legitimate in the strip $\Re\mathrm{e}\,\omega < 2$ of the complex $\omega-$plane. Expanding around $2-\omega \equiv \epsilon\,,\;0<\epsilon<1\,,$ we find

$$
\begin{aligned}
\Gamma(-n+\epsilon) &= \frac{(-1)^n}{n!}\left\{\frac{1}{\epsilon}+\psi(n+1)\right. \\
&+ \left.\frac{\epsilon}{2}\left[\frac{\pi^2}{3}+\psi^2(n+1)-\psi'(n+1)\right]\right\} \\
&+ O\left(\epsilon^2\right) \qquad n=0,1,2,\,\ldots \qquad (4.23)
\end{aligned}
$$

where

$$\psi(z) = \frac{d}{dz}\,\ln\Gamma(z)$$

$$\psi(n+1) = 1+\frac{1}{2}+\frac{1}{3}+\,\cdots\,+\frac{1}{n}-\mathbf{C}$$

$$\psi'(n+1) = \frac{\pi^2}{6}+\sum_{\jmath=1}^{n}\,\jmath^{-2} \qquad [\,\psi'(1)=\pi^2/6\,]$$

$\mathbf{C}$ being the Euler$-$Mascheroni constant

$$\psi(1) = -\,\mathbf{C} = -\,0.5772\ldots$$

Hence we finally come to the expansion

$$
\begin{aligned}
I &= \frac{1}{16\pi^2}\left(\frac{1}{\epsilon}-\,\mathbf{C}\;+\cdots\right)\left(1+\epsilon\,\ln\frac{4\pi\mu^2}{\Delta}\;+\cdots\right) \\
&= \frac{1}{16\pi^2}\left(\frac{1}{\epsilon}-\,\mathbf{C}+\ln\frac{4\pi\mu^2}{\Delta}\right)+O(\epsilon) \qquad (4.24)
\end{aligned}
$$

which shows that the divergence can be segregated as a simple pole in the so named $\epsilon-$expansion.

Consider now the so called *sweetmeat diagram* which corresponds to any of the three diagrams contributing to the 4$-$point connected Green's functions in momentum space

$$J_{12}(k) \;=\; \frac{1}{2}\,(-\,\mathrm{i}\lambda)^2 \int \frac{d\ell}{(2\pi)^4}\,\frac{i}{\ell^2 - m^2 + \mathrm{i}\varepsilon}\,\frac{i}{(\ell - k)^2 - m^2 + \mathrm{i}\varepsilon} \tag{4.25}$$

where $k = k_1 + k_2$ and after setting for the sake of brevity

$$\int_\ell \stackrel{\mathrm{def}}{=} \mu^{2\epsilon} \int \frac{d^{2\omega}\ell}{(2\pi)^{2\omega}} \qquad [\,\epsilon = 2 - \omega\,]$$

we come to the dimensionally regularized Feynman integral

$$J_s(k) \;=\; \frac{\lambda^2}{2} \int_\ell \frac{1}{(\ell^2 - m^2 + \mathrm{i}\varepsilon)[\,(\ell - k)^2 - m^2 + \mathrm{i}\varepsilon\,]} \tag{4.26}$$

Using Feynman parametrization formula we get

$$J_s(k) \;=\; \frac{\lambda^2}{2} \int_0^1 dx \int_\ell \frac{1}{[\,\ell^2 - m^2 - 2\ell\cdot k\,(1-x) + k^2\,(1-x) + \mathrm{i}\varepsilon\,]^2}$$

and after a shift of the integration variable

$$\ell' = \ell - (1-x)\,k$$

we end up with

$$J_s(k) \;=\; \frac{\lambda^2}{2} \int_0^1 dx \int_\ell \frac{1}{[\,\ell^2 - m^2 + x(1-x)k^2 + \mathrm{i}\varepsilon\,]^2}$$

Now we can perform the Wick rotation with $p_0 = ip_4$ and get

$$J_s(k) \;=\; \frac{i\lambda^2}{2} \int_0^1 dx \int_\ell \frac{1}{[\,\ell^2 + m^2 + x(1-x)k_E^2\,]^2}$$

with $(-\,k^2) = \mathbf{k}^2 + k_4^2 = k_E^2 > 0$. Taking the Mellin transform

$$\begin{aligned} J_s(k) \;&=\; \frac{i\lambda^2}{2} \int_0^1 dx \int_0^\infty t\,dt\;\exp\{-\,t\,[\,m^2 + x(1-x)k_E^2\,]\} \int_\ell \exp\{-\,t\,\ell^2\} \\ &=\; \frac{i\lambda^2}{2} \int_0^1 dx \int_0^\infty t^{1-\omega}\,dt\;\exp\{-\,t\,[\,m^2 + x(1-x)k_E^2\,]\}\,\frac{\mu^{2\epsilon}}{(4\pi)^\omega} \end{aligned}$$

$$
\begin{aligned}
&= \frac{i\lambda^2}{32\pi^2}\,\Gamma(\epsilon)\int_0^1 dx\;\left[\frac{4\pi\mu^2}{m^2+x(1-x)\,k_E^2}\right]^\epsilon \\
&= \frac{i\lambda^2}{32\pi^2}\left\{\frac{1}{\epsilon}-\mathbf{C}+\int_0^1 dx\;\ln\frac{4\pi\mu^2}{m^2-x(1-x)\,k^2}\right\}+O(\epsilon) \\
&= \frac{i\lambda^2}{32\pi^2}\left\{\frac{1}{\epsilon}-\mathbf{C}+\ln\frac{4\pi\mu^2}{k^2}-\int_0^1 dx\;\ln R\right\}+O(\epsilon) \\
&= \frac{i\lambda^2}{32\pi^2}\left\{\frac{1}{\epsilon}-\mathbf{C}+\ln\frac{4\pi\mu^2}{m^2}+\int_0^1 \frac{dx}{R}\;(2x^2-x)\right\}+O(\epsilon)
\end{aligned}
$$

where I have set as before $R = x^2 - x + m^2/k^2$. Now for $0 < k^2 < 4m^2$ we obtain

$$
\begin{aligned}
J_s(k) &= \frac{i\lambda^2}{32\pi^2}\left\{\frac{1}{\epsilon}-\mathbf{C}+\ln\frac{4\pi\mu^2}{m^2}+2\left(1-\sqrt{\Delta}\,\mathrm{arcctg}\sqrt{\Delta}\right)\right\} \\
&= \frac{i\lambda^2}{32\pi^2}\left\{\frac{1}{\epsilon}-\mathbf{C}+\ln\frac{4\pi\mu^2}{m^2}+2\right. \\
&\quad \left. -\sqrt{\frac{4m^2}{k^2}-1}\quad\mathrm{arcctg}\sqrt{\frac{4m^2}{k^2}-1}\;\right\}+O(\epsilon) \qquad (4.27)
\end{aligned}
$$

We have to remember that in the evaluation of the 4−point connected Green's function for the self−interacting real scalar field theory, there will be three such contributions with

$$
k = k_1 + k_2\,, \qquad k = k_1 + k_3\,, \qquad k = k_1 + k_4
$$

corresponding to the $s-$, $t-$ and $u-$channels respectively[2].

4.1.4 Vacuum polarization tensor in QED

Consider the photon self−energy diagram in quantum electrodynamics that gives rise to the vacuum polarization tensor which is defined to be

$$
\mathtt{reg}\,\Pi^{\mu\nu}(k,M,\mu) = -ie^2\int_p \mathrm{tr}\,\gamma^\mu S_F(p,M)\gamma^\nu S_F(p+k,M) \qquad (4.28)
$$

where

$$
\int_p \stackrel{\mathrm{def}}{=} \mu^{2\epsilon}\int\frac{d^{2\omega}p}{(2\pi)^{2\omega}} \qquad [\,\epsilon = 2-\omega\,]
$$

[2] *Nota Bene* : all momenta are incoming.

Taking the traces

$$\mathrm{tr}\,\gamma^{\mu}\gamma^{\nu} \;=\; g^{\mu\nu}\,\mathrm{tr}\,\mathbb{I} \;=\; 2^{\,\omega}\,g^{\mu\nu} \tag{4.29}$$

$$\mathrm{tr}\,\gamma^{\kappa}\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu} \;=\; 2^{\,\omega}\left(g^{\kappa\lambda}\,g^{\mu\nu} \;-\; g^{\kappa\mu}\,g^{\lambda\nu} \;+\; g^{\kappa\nu}\,g^{\lambda\mu}\right) \tag{4.30}$$

we readily come to the expression

$$\begin{aligned}
\mathtt{reg}\,\Pi^{\mu\nu}(k,M,\mu) \;&=\; 2^{\,\omega}\,ie^2\int_p \frac{2\,p^{\mu}p^{\nu}+p^{\mu}k^{\nu}+p^{\nu}k^{\mu}}{(p^2-M^2+\mathrm{i}\varepsilon\,)\,[\,(p+k\,)^2-M^2+\mathrm{i}\varepsilon\,]}\\
&-\;2^{\,\omega}\,ie^2\int_p \frac{g^{\mu\nu}\,[\,p\cdot(p+k)-M^2\,]}{(p^2-M^2+\mathrm{i}\varepsilon\,)\,[\,(p+k\,)^2-M^2+\mathrm{i}\varepsilon\,]}\\
&=\;2^{\,\omega}\,ie^2\int_p \frac{2\,p^{\mu}p^{\nu}+p^{\mu}k^{\nu}+p^{\nu}k^{\mu}-p\cdot k\,g^{\mu\nu}}{(p^2-M^2+\mathrm{i}\varepsilon\,)\,[\,(p+k\,)^2-M^2+\mathrm{i}\varepsilon\,]}\\
&-\;2^{\,\omega}\,i\,e^2\,g^{\mu\nu}\int_p\,(\,p^{\,2}-M^{\,2}+\mathrm{i}\varepsilon\,)^{-1}
\end{aligned} \tag{4.31}$$

The very last integral can be obtained from the generating integral

$$I_n(Q)\;\stackrel{\mathrm{def}}{=}\;\int_p\,(\,p^2-Q+\mathrm{i}\varepsilon\,)^{-n} \qquad (\,Q\ge 0\,,\;\;n\in\mathbb{N}\,) \tag{4.32}$$

which is convergent in a $D-$dimensional space$-$time with $D=\Re\mathrm{e}\,\omega<n\,.$ To calculate this integral, let me perform the Wick rotation, that is, let us consider the closed oriented contour γ^{+} in the complex energy$-$plane of fig. N 12. Notice that, thanks to the causal prescription, the two poles of the integrand

$$p_0 \;=\; \begin{cases} \;\;\sqrt{\mathbf{p}^{\,2}+Q}-i\,\varepsilon \\ -\sqrt{\mathbf{p}^{\,2}+Q}+i\,\varepsilon \end{cases}$$

lie outside the contour γ^{+}. Since the contributions of the two large arcs do vanish when $R\;\to\;\infty$ we obtain

$$\begin{aligned}
I_n(Q)\;&\stackrel{\mathrm{def}}{=}\;i\,(-1)^n\int_{p_E}\,(\,p_E^2+Q)^{-n} \qquad (\,Q\ge 0\,,\;\;n\in\mathbb{N}\,)\\
&=\;\frac{i\,\mu^{4-2\omega}\,(-1)^n}{(2\pi)^{2\omega}\,\Gamma\,(n)}\int_0^{\infty} dt\;t^{\,n-1}\;\mathrm{e}^{-\,tQ}\int d^{2\omega}p_E\;\mathrm{e}^{-\,tp_E^2}\\
&=\;\frac{i\,\mu^{4-2\omega}\,(-1)^n}{(4\pi)^{\omega}\,\Gamma\,(n)}\int_0^{\infty} dt\;t^{\,n-\omega-1}\;\mathrm{e}^{-\,tQ}\\
&=\;i\;\frac{Q^{\,2-n}}{16\pi^{\,2}}\;(-1)^n\,\frac{\Gamma\,(n-\omega)}{\Gamma\,(n)}\;\left(\frac{4\pi\mu^2}{Q}\right)^{\epsilon}
\end{aligned} \tag{4.33}$$

As a consequence, for $n = 1$ we obtain

$$\begin{aligned}\int_p \frac{1}{p^2 - M^2 + \mathrm{i}\varepsilon} &= \frac{-\,\mathrm{i}\,M^2}{16\,\pi^2}\,\Gamma(-1+\epsilon)\left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon} \\ &= \frac{i\,M^2}{16\,\pi^2}\left[\frac{1}{\epsilon} + \psi\,(2) + \cdots\right]\left(1 + \epsilon\,\ln\frac{4\pi\mu^2}{M^2} + \cdots\right) \\ &= \frac{i\,M^2}{16\,\pi^2}\left[\frac{1}{\epsilon} + \psi\,(2) - \ln\frac{M^2}{4\pi\mu^2} + \cdots\right] \qquad (4.34)\end{aligned}$$

To calculate the remaining integrals it is convenient to introduce the corresponding generating integrals. For example, let me first consider the general scalar integral

$$\begin{aligned}I_{r\,,\,s}\,(k, M\,;\mu,\omega) &= \int_p \left(p^2 - M^2 + i\varepsilon\right)^{-r}\left[\,(p+k)^2 - M^2 + i\varepsilon\,\right]^{-s} \\ &= \frac{\Gamma\,(r+s)}{\Gamma\,(r)\,\Gamma\,(s)}\int_0^1 dx\;x^{\,r-1}\,(1-x)^{\,s-1}\,\times \\ &\times\;\int_p\left[\,p^2 - M^2 + i\varepsilon + 2p\cdot k\,(1-x) + k^2\,(1-x)\,\right]^{-r-s}\end{aligned}$$

A translation of the integration variable in momentum space drives to

$$\begin{aligned}&p^{\mu} \;=\; p'^{\mu} - (1-x)\,k^{\mu} \\ &p^2 - M^2 + 2p\cdot k(1-x) + k^2(1-x) \;= \\ &p'^{\,2} - M^2 + x(1-x)\,k^2 = p'^{\,2} - k^2\,R\,(x\,,\,a) \qquad (4.35)\end{aligned}$$

in which I make use of the already introduced notations

$$R\,(x\,,\,a) \;\stackrel{\mathrm{def}}{=}\; x^2 - x + M^{\,2}\,/\,k^{\,2} \;=\; a - x + x^2$$

$$\Delta \;\stackrel{\mathrm{def}}{=}\; \frac{4\,M^{\,2}}{k^{\,2}} - 1 \;=\; 4a - 1$$

Consider now the integral

$$\int_p\,[\,p^{\,2} - k^2\,R\,(x\,,\,a)\,]^{-r-s} \qquad (4.36)$$

After performing the Wick rotation [3] we end up with the family of parametric integrals

$$\begin{aligned}I_{r\,,\,s}\,(k, M\,;\mu,\omega) &= \frac{i}{16\pi^2}\,(-1)^{r+s}\,(4\pi\mu^2)^{2-\omega}\,\frac{\Gamma\,(r+s-\omega)}{\Gamma\,(r)\,\Gamma\,(s)} \\ &\times\;\int_0^1 dx\;x^{r-1}\,(1-x)^{s-1}\,[\,k^2\,R\,(x\,,\,a)\,]^{\omega-r-s}\end{aligned}$$

[3] See *e.g.* M.E. Peskin & D.V. Schroeder, *An Introduction to Quantum Field Theory*, Perseus Books, Reading (1995) equation (A44) p. 807.

which are elementary and can be calculated in a straightforward although tedious way after partial integrations in terms of the well known basic integral and subsequent list, as repoted in the Appendix. The final result is

$$\texttt{reg}\,\Pi^{\mu\nu}(k,M,\mu) \;=\; \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) \texttt{reg}\,\Pi(k^2, M^2) \qquad (4.37)$$

in which the invariant scalar polarization function is provided by

$$\begin{aligned}
\texttt{reg}\,\Pi(k^2, M^2) &= \frac{-8e^2}{(4\pi)^\omega}\,\mu^{2\epsilon}\int_0^1 dx\;\frac{x(1-x)\,\Gamma(2-\omega)}{[\,M^2 - x(1-x)\,k^2\,]^\epsilon} \\
&= \frac{-2\alpha}{\pi}\int_0^1 dx\;x(1-x)\,\Gamma(\epsilon)\left[\frac{4\pi\mu^2}{M^2 - x(1-x)\,k^2}\right]^\epsilon \\
&= \frac{-2\alpha}{\pi}\int_0^1 dx\;x(1-x)\;\times \\
&\times\;\left\{\frac{1}{\epsilon} - \mathbf{C} - \ln\left[\frac{M^2 - x(1-x)k^2}{4\pi\mu^2}\right] + \mathrm{O}(\epsilon)\right\} \\
&= \frac{-\alpha}{3\pi}\left\{\frac{1}{\epsilon} - \mathbf{C} + \ln\frac{4\pi\mu^2}{M^2}\right. \\
&- \left.\int_0^1 \frac{dx}{R}\left(4x^4 - 8x^3 + 3x^2\right)\right\} \;+\; O(\epsilon)
\end{aligned}$$

Notice that In the neighbourhood of the photon mass shell $k^2 = 0$, that means in the vicinity of the light−cone, we have the behaviour

$$\frac{-\alpha}{3\pi}\int_0^1 \frac{dx}{R}\left(4x^4 - 8x^3 + 3x^2\right) \sim \frac{\alpha\,k^2}{15M^2} \qquad\qquad (k^2 \to 0)$$

It follows that the finite part of the invariant polarization function reads

$$\begin{aligned}
\widehat{\Pi}(k^2, M^2) &\stackrel{\mathrm{def}}{=} \Pi(k^2, M^2) - \Pi(0, M^2) \\
&= \frac{-2\alpha}{\pi}\int_0^1 dx\;x(1-x)\;\ln\frac{M^2}{Rk^2} = \frac{-2\alpha}{\pi}\,\bar{I}_2 \qquad (4.38)
\end{aligned}$$

From ref. [21] eq. **2.172** p. 81, eq.s **2.174**1. p. 82 with

$$a = \frac{M^2}{k^2} < 0 \qquad b = -1 \qquad c = 1 \qquad \Delta = \frac{4M^2}{k^2} - 1 < 0$$

for $0 < k^2 < 4M^2$ we get

$$\widehat{\Pi}(k^2, M^2) \;=\; \frac{\alpha}{3\pi}\int_0^1 \frac{dx}{R}\left(4x^4 - 8x^3 + 3x^2\right) \;=\; -\frac{2\alpha}{\pi}\,\bar{I}_2$$

$$
\begin{aligned}
&= \frac{-\alpha}{3\pi}\left\{\frac{5}{3}+\frac{4M^2}{k^2}-\left[\frac{M^2}{k^2}\left(1+\frac{4M^2}{k^2}\right)-\frac{1}{2}\right]\int_0^1\frac{dx}{R}\right\}\\
&= \frac{\alpha}{3\pi}\left\{\frac{1}{3}+2\left(1+\frac{2M^2}{k^2}\right)\left[\left(\frac{4M^2}{k^2}-1\right)^{1/2}\right.\right.\\
&\times \ \operatorname{arcctg}\left.\left.\left(\frac{4M^2}{k^2}-1\right)^{1/2}-1\right]\right\}
\end{aligned}
\tag{4.39}
$$

in agreement with [12] § **7-1-1** eq. (7-9) p. 323. Now, in order to unravel the analytic structure of the invariant polarization function it is convenient to come back to the integral representation

$$
\widehat{\Pi}(k^2, M^2) = \frac{2\alpha}{\pi}\int_0^1 dx\ x(1-x)\ \ln\frac{k^2 R}{M^2}
$$

After setting $k^2 = \Re\mathrm{e}\, s$, it appears that the integrand is real and analytic for $k^2 < 4M^2 \vee 0 \le x \le 1$, while it exhibits a branch point when the argument of the logarithm does vanish and a cut in the complex $s-$plane for

$$
R(x,a) = x^2 - x + \frac{M^2}{k^2} < 0 \tag{4.40}
$$

that is

$$
\Re\mathrm{e}\, s = k^2 > 4M^2 \qquad \frac{1}{2}-\frac{\beta}{2} < x < \frac{1}{2}+\frac{\beta}{2} \tag{4.41}
$$

$$
\beta \equiv \sqrt{1-\frac{4M^2}{k^2}} \qquad [\,0 \le \beta < 1\,]
$$

As a consequence, the imaginary part of the the polarization function can be readily obtained above/below the cut by

$$
\ln\left(x^2 - x + \frac{M^2}{k^2 \pm \mathrm{i}0}\right) = \ln(-R) \mp i\pi \qquad (R<0)
$$

which yields

$$
\begin{aligned}
\Im\mathrm{m}\,\widehat{\Pi}(k^2 \pm i0, M^2) &= \mp 2\alpha\int_{(1-\beta)/2}^{(1+\beta)/2} dx\ x(1-x)\\
&= \mp\frac{\alpha}{3}\left(1+\frac{2M^2}{k^2}\right)\sqrt{1-\frac{4M^2}{k^2}}
\end{aligned}
\tag{4.42}
$$

in accordance [4] with [19] § **7.5** eq. (7.92) p. 253 and [12] § **7-1-1** eq. (7-11) p. 323. It turns out that the discontinuity across the cut

$$\texttt{reg}\,\Pi(\,k^2+i\,0\,,\,M^2\,)-\texttt{reg}\,\Pi(\,k^2-i\,0\,,\,M^2\,)=2i\,\Im\mathrm{m}\,\widehat{\Pi}\,(\,k^2+i\,0\,,\,M^2\,)$$

does not depend upon regularization, *i.e.* it is finite, and has exactly the same energy−dependence, up to the subsitution $k^2 \leftrightarrow 4E$, of the cross section (2.74) for the production of a fermion−antifermion pair, the parameter β being precisely the fermion velocity in the center of momentum frame.

Exercise : let me calculate with the dimensional regularisation the 1-loop pion self−energy of the Yukawa theory and compare the result with the Pauli−Villars regularisation of eq. (4.20). According to the expression

$$\begin{aligned}\texttt{reg}\,\Sigma(k\,;\,\mu,\omega\,) &\stackrel{\mathrm{def}}{=} \mathrm{i}g^2\,\mu^{4-2\omega}\,(2\pi)^{-2\omega}\int \mathrm{d}^{2\omega}p\;\mathrm{tr}\,S_F\,(\,p\,)\,S_F\,(\,p+k\,)\\ &= -\,4\mathrm{i}g^2\int_p \frac{p^2+M^2+p\cdot k}{(p^2-M^2+\mathrm{i}\varepsilon)[\,(p+k)^2-M^2+\mathrm{i}\varepsilon\,]}\end{aligned}$$

By making use of the list of integrals in the Appendix we readily get

$$\begin{aligned}\texttt{reg}\,\Sigma(k\,;\,\mu,\omega\,) &= -\,4\mathrm{i}g^2\,\{g^{\mu\nu}\,I_{\mu\nu}(1,1)-k^\mu\,I_\mu(1,1)+M^2\,I(1,1)\}\\ &\doteq \frac{3g^2M^2}{4\pi^2}\left(\frac{1}{\epsilon}-\mathbf{C}+I_0+\frac{2}{3}\right)\\ &- \frac{g^2k^2}{8\pi^2}\left(\frac{1}{\epsilon}-\mathbf{C}+6I_2+\frac{2}{3}\right)\\ &= \frac{3g^2}{4\pi^2}\,(M^2-k^2/6)\left(\frac{1}{\epsilon}-\mathbf{C}+\ln\frac{4\pi\mu^2}{M^2}+\frac{2}{3}\right)\\ &- \frac{3g^2}{4\pi^2}\,(\bar{I}_2\,k^2-\bar{I}_0\,M^2)\\ &= \frac{3g^2}{4\pi^2}\,(M^2-k^2/6)\left(\frac{1}{\epsilon}-\mathbf{C}+\ln\frac{4\pi\mu^2}{M^2}\right)-\frac{7g^2k^2}{24\pi^2}\\ &+ \frac{3g^2M^2}{2\pi^2}-\frac{g^2k^2}{4\pi^2}\left(\frac{4M^2}{k^2}-1\right)^{3/2}\mathrm{arcctg}\left(\frac{4M^2}{k^2}-1\right)^{1/2}\end{aligned}$$

It is important to realize that the sign of the divergent part, as well as the whole non-polynomial part, *i.e.* the very last term

$$-\,\frac{g^2k^2}{4\pi^2}\left(\frac{4M^2}{k^2}-1\right)^{3/2}\mathrm{arcctg}\left(\frac{4M^2}{k^2}-1\right)^{1/2}$$

exacly coincide with the corresponding quantities (4.20) which has been obtained in the Pauli−Villars regularisation. In other words, it turns out that the arbitrariness in the finite part of the above 1−loop regularized quantity does merely concern the polynomial part in momentum space, that is the local part in configuration space. This feature will represent, as we shall see futher on, the key point of the renormalisation procedure.

[4] Notice however that the two textbooks use opposite signs, *i.e.* $\bar{\omega}(k^2,m,\Lambda)=-\,\Pi_2(q^2)$.

4.1.5 Vacuum polarization effects

Next let us examine how the finite part of the invariant polarization function $\widehat{\Pi}\,(k^2, M^2)$ does modify the electromagnetic interaction. Actually, it turns out that in the non−relativistic limit it makes sense to compute the potential $V(r)$, that will contain the modifications to the classical Coulomb potential caused by the Heisenberg and the relativity principles : the emission and absorption of virtual pairs, that is the vacuum polarization effect. Let me recall that for two incoming and two outgoing distinguishable particles of equal mass M but different charges $-e$ and $-Ze$ respectively, the leading order contribution to the scattering amplitude is given by eq. (1.40)

$$\bar{u}_{r'}(p')\,(ie\gamma^{\mu})\,u_r(p)\;\frac{-\,\mathrm{i}\,g_{\mu\nu}}{(p-p')^2}\;\bar{u}_{s'}(q')\,(iZe\gamma^{\nu})\,u_s(q)$$

Once again, in the non−relativistic limit

$$\bar{u}_{r'}(p')\,\gamma^0\,u_r(p)\;\approx\;2M\,\delta_{rr'}\qquad\text{et cetera}$$

where M is the particle mass in such a manner that we can write

$$\frac{-\,\mathrm{i}Ze^2}{|\,\mathbf{p}-\mathbf{p}'\,|^2}\,2M\,\delta_{rr'}\,2M\,\delta_{ss'} = i\,T_{\mathbf{p},\mathbf{p}'}\,2M\,\delta_{rr'}\,\delta_{ss'}$$

and consequently

$$T_{\mathbf{p},\mathbf{p}'} = f(\theta) = -\,\frac{2MZe^2}{|\,\mathbf{p}-\mathbf{p}'\,|^2}$$

which corresponds to the repulsive Coulomb potential

$$V(r) = \frac{Ze^2}{4\pi r} = Z\,\frac{\alpha}{r}$$

so that

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \frac{Z^2\alpha^2}{4M^2v^4\,\sin^4(\theta/2)}\qquad(\mathbf{p}=M\mathbf{v})$$

which is nothing but the celebrated Rutherford classical cross section.

Now, to the aim of taking into account the radiative corrections in the non−relativistic limit, I can write in analogy

$$\begin{aligned}\widehat{V}(r) &= \lim_{\mu\to0}\;Ze^2\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\,\frac{\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2+\mu^2}\left[1-\widehat{\Pi}\,(-\,\mathbf{k}^2, M^2)\right]\\ &= \lim_{\mu\to0}\;\frac{-\,iZe^2}{4\pi^2 r}\int_{-\infty}^{\infty}\mathrm{d}k\;\frac{k\,\mathrm{e}^{\mathrm{i}kr}}{k^2+\mu^2}\;\left[1-\widehat{\Pi}\,(-\,k^2, M^2)\right]\qquad(4.43)\end{aligned}$$

where I have introduced the small photon mass μ as an infrared regulator for the Coulomb potential. To calculate this integral, we consider the complex $k-$plane and a big half−circle in the upper half−plane centered at the origin, with diameter on the real axis and very large ray $R \to \infty$. Notice however the the upper half−plane has a cut starting from $\Im\mathrm{m}\, k = 2M$ to infinity, for the invariant polarization function has a branch point at $-k^2 = (\mathrm{i}k)^2 = 4M^2$, as I have discussed before. Furthermore there is a simple pole at $k = i\mu$, leading to the Coulomb potential after removal of the infrared regulator μ. Since the real part of $\widehat{\Pi}(-k^2, M^2)$ takes the same value on both sides of the cut, it follows that the modifications to the Coulomb potential are solely due to the imaginary part of the invariant polarization function, *i.e.* to its discontinuity. Hence we readily obtain from equation (4.42)

$$\begin{aligned} \delta\,\widehat{V}(r) &= \frac{Ze^2}{2\pi^2 r}\int_{2M}^{\infty} \mathrm{d}k\ \frac{\mathrm{e}^{-kr}}{k}\ \Im\mathrm{m}\,\widehat{\Pi}(k^2 - i0, M^2) \\ &= \frac{2Z\alpha^2}{3\pi r}\int_{2M}^{\infty} \mathrm{d}k\ \frac{\mathrm{e}^{-kr}}{k}\left(1 + \frac{2M^2}{k^2}\right)\sqrt{1 - \frac{4M^2}{k^2}} \end{aligned} \tag{4.44}$$

At large distances when $r \gtrsim 1/M$ this integral is dominated by the region where $k \sim 2M$. Approximating there the integrand function and changing the integration variable according to $t = k - 2M$ we find

$$\begin{aligned} \delta\,\widehat{V}(r) &= \frac{2Z\alpha^2}{3\pi r}\int_0^{\infty} \mathrm{d}t\ \frac{\mathrm{e}^{-r(t+2M)}}{2M}\ \frac{3}{2}\sqrt{\frac{t}{M}} + \cdots \\ &\approx \frac{Z\alpha^2}{4r\sqrt{\pi}}\,(Mr)^{-3/2}\,\mathrm{e}^{-2Mr} \end{aligned} \tag{4.45}$$

so that, in conclusion, the modified Coulomb potential approximately reads

$$\widehat{V}(r) = \frac{Z\alpha}{r}\left[1 + \frac{\alpha}{4\sqrt{\pi}}\,(Mr)^{-3/2}\,\mathrm{e}^{-2Mr}\right] + \cdots \tag{4.46}$$

Thus we see that the range of the correction term is of the order of the Compton wavelength $\hbar/Mc$ of the particles. The radiative correction to the Coulomb potential is named the *Serber−Uehling potential*

Robert Serber
Linear Modifications in the Maxwell Field Equations
Physical Review **48** (1935) 49 - 54 [Issue 1 – July 1935]
Edwin A. Uehling
Polarization Effects in the Positron Theory
Physical Review **48** (1935) 55 - 63 [Issue 1 – July 1935]

We can interpret the result as being due to screening. When the two point-like charges, say two electrons, are at the distance of the electron Compton wavelength $\lambda_e = \hbar / m_e c = 3.861\,592\,678(26) \times 10^{-13}$ m, the emission and absorption of virtual $e^- e^+$ pairs make the vacuum a dielectric medium in which the apparent charge is less than the *bare charge* e_0 . At shorter distance we begin to penetrate the polarization cloud and see the bare charge, which is bigger and bigger as far as we penetrate closer and closer. This is known as the vacuum polarization effect.

Conversely, in the very small distances limit $\mathbf{k}^2 = -k^2 \gg M^2$ we can safely approximate

$$
\begin{aligned}
\widehat{\Pi}(k^2, M^2) &= -\frac{2\alpha}{\pi} \int_0^1 dx\, x(1-x) \ln\left[\frac{M^2 - x(1-x)k^2}{M^2}\right] \\
&\approx -\frac{2\alpha}{\pi} \int_0^1 dx\, x(1-x) \left\{ \ln\left(-\frac{k^2}{M^2}\right) + \ln[\,x(1-x)\,] \right\} \\
&= -\frac{\alpha}{3\pi} \left\{ \ln\left(\frac{-k^2}{\mathrm{e}^{5/3} M^2}\right) + O(M^2/k^2) \right\}
\end{aligned}
\tag{4.47}
$$

As a consequence the effective electric coupling in the limit of very small distances becomes approximately

$$
\begin{aligned}
\alpha_{\text{eff}}(k, M) &\approx \alpha \left[1 + \frac{\alpha}{3\pi} \ln\left(\frac{-k^2}{\mathrm{e}^{5/3} M^2}\right) \right] \\
&\approx \alpha \left[1 - \frac{\alpha}{3\pi} \ln\left(\frac{-k^2}{\mathrm{e}^{5/3} M^2}\right) \right]^{-1}
\end{aligned}
\tag{4.48}
$$

Of course, the above approximate short−distance behaviour of the effective charge can be trusted as long as

$$
1 - \frac{\alpha}{3\pi} \ln\left(\frac{-k^2}{\mathrm{e}^{5/3} M^2}\right) \gg \alpha \tag{4.49}
$$

which leads to a singularity for

$$
|\,k^2\,| = M'^2\, \mathrm{e}^{3\pi/\alpha} \qquad\qquad M' = \mathrm{e}^{5/6} M
$$

the famous Landau−Pomerančuk[5] singularity. However, well before we reach such a large scale, the perturbative approximate equality (4.48) has to be amended by higher order corrections which can no longer be neglected.

The combined vacuum polarization effects for $e^- e^+$ plus heavier charged leptons and quarks makes the value of $\alpha_{\text{eff}}(k)$ to increase by about 5%

[5] L.D. Landau and I. Ja. Pomeranchuk, *Doklady Akad. Nauk USSR*, **102** (1955) 489.

from $k = 0$ to $k = 30$ GeV, as observed in high−energy experiments, with $\alpha_{\rm eff}\,(0) \equiv \alpha$. The idea of a distance−dependent, or scale−dependent or even *running* coupling parameter is the main result of the *renormalization group invariance* of perturbative renormalizable quantum field theories, as will be better focussed below.

Exercise : calculate the contour integral leading to the Serber-Uehling radiative correction $\delta\,V(r)$ to the classical Coulomb potential. To this purpose, consider the functions of the complex variable $z = x + {\rm i}y$

$$f(z) = \frac{z\,{\rm e}^{\,{\rm i}\,z\,r}}{z^2 + \mu^2}\;g\,(z) \qquad\qquad r > 0$$

$$g(z) = \int_0^1 dx\;x\,(1-x)\;\ln\left[\frac{-\,z^2}{M^2}\left(x^2 - x - \frac{M^2}{z^2}\right)\right]$$

in such a manner that we have

$$\delta\,V(r) = Z\,\frac{2\pi}{3r}\,\left(\frac{\alpha}{\pi}\right)^2\int_{-\infty}^{\infty}{\rm d}x\;f(x)$$

The complex function $g(z)$ exhibits a branch point in the upper half-plane at $y = 2M$, leading to a cut along the positive imaginary axis from $2M$ to infinity. It turns out that the real part $\Re{\rm e}\,g(z)$ is continuous across the cut, while the imaginary part $\Im{\rm m}\,g(z)$ has a discontinuity across the cut which is given by

$$\Im{\rm m}\,g(0^+ + {\rm i}y) - \Im{\rm m}\,g(0^- + {\rm i}y) = \left(1 - \frac{2M^2}{z^2}\right)\sqrt{1 + \frac{4M^2}{z^2}} \qquad\quad (y > 2M)$$

Consider now the oriented contour γ^+ of fig. N 13 so that

$$\oint_{\gamma^+} f(z)\;{\rm d}z = 2\pi{\rm i}\;\lim_{\zeta\,\to\,{\rm i}\mu}\;(\zeta - {\rm i}\mu)\,f(\zeta) \tag{4.50}$$

The contributions from the two large arcs ($z = R{\rm e}^{{\rm i}\,\theta}$, $\eta\;\to\;0^+$) yield

$$\begin{aligned}&{\rm i}R^2\left(\int_0^{\pi/2-\eta} + \int_{\eta+\pi/2}^{\pi}\right)(R^2 + \mu^2{\rm e}^{-2i\theta})^{-1}\;\exp\{\,{\rm i}rR\cos\theta - rR\sin\theta\,\}\\ &\times\left\{\tfrac{1}{6}\;\ln\,\frac{R^2}{M^2} + \int_0^1 dx\;x\,(1-x)\;\ln\,\left[\frac{M^2}{R^2} - x(1-x)\,{\rm e}^{2i\theta}\right]\right\}{\rm d}\theta\;\stackrel{R\to\infty}{\longrightarrow}\;0\end{aligned} \tag{4.51}$$

which rapidly vanish when $R\;\to\;\infty$. Similarly, the contribution from the small circle around the branch point ($z = 2iM + \rho\,{\rm e}^{{\rm i}\phi}$, $\eta\;\to\;0^+$) does vanis, *viz.,*

$${\rm i}\,\rho\int_{\eta+\pi/2}^{\pi/2-\eta}{\rm d}\phi\;{\rm e}^{i\phi}\;f\left(2{\rm i}M + \rho\,{\rm e}^{i\phi}\right)\;\stackrel{\rho\,\to\,0}{\longrightarrow}\;0 \tag{4.52}$$

For the contributions along the cut ($z = \pm\,\eta + {\rm i}y$, $\mu < 2M < y < R$, $\eta\;\to\;0^+$) we have

$${\rm i}\int_{2M}^{R}{\rm d}y\;\frac{y\,{\rm e}^{-\,ry}}{\mu^2 - y^2}\;\left[\,g({\rm i}y + 0^-)\;-\;g({\rm i}y + 0^+)\,\right] \tag{4.53}$$

It turns out that

$$\Re\mathrm{e}\big[\, g(\mathrm{i}y + 0^-) \;-\; g(\mathrm{i}y + 0^+)\,\big] = 0 \qquad\qquad (\,y > 2M\,)$$

while

$$\Im\mathrm{m}\big[\, g(0^+ + \mathrm{i}y) - g(0^- + \mathrm{i}y)\,\big] = \left(1 + \frac{2M^2}{y^2}\right)\sqrt{1 - \frac{4M^2}{y^2}} \qquad (y > 2M)$$

Putting altogether we eventually obtain

$$\begin{aligned}
\lim_{R\to\infty}\lim_{\rho\to 0}\oint_{\gamma+} f(z)\,\mathrm{d}z &= \int_{-\infty}^{\infty}\mathrm{d}x\; f(x) \\
&+ \int_{2M}^{\infty}\mathrm{d}y\;\frac{y\,\mathrm{e}^{-\,ry}}{\mu^2 - y^2}\;\left(1 + \frac{2M^2}{y^2}\right)\sqrt{1 - \frac{4M^2}{y^2}} \\
&= \pi\mathrm{i}\;\mathrm{e}^{-\,\mu r}\int_0^1 dx\; x\,(1-x)\;\ln\left[\frac{\mu^2}{M^2}\left(x^2 - x + \frac{M^2}{\mu^2}\right)\right]
\end{aligned}$$

in such a manner that in the limit $\mu \to 0$ we finally get Serber−Uehling correction to the classical Coulomb potential

$$\begin{aligned}
\delta\,V(r) &= Z\,\frac{2\pi}{3r}\,\left(\frac{\alpha}{\pi}\right)^2\;\lim_{\mu\to 0}\int_{-\infty}^{\infty}\mathrm{d}x\; f(x) \\
&= Z\,\frac{2\pi}{3r}\,\left(\frac{\alpha}{\pi}\right)^2\int_{2M}^{\infty}\frac{\mathrm{d}y}{y}\;\mathrm{e}^{-\,ry}\;\left(1 + \frac{2M^2}{y^2}\right)\sqrt{1 - \frac{4M^2}{y^2}}
\end{aligned} \tag{4.54}$$

in accordance with eq. (4.44).

4.1.6 Effective action

Consider the generating functional of the connected Green's functions for the free scalar field theory, *i.e.*

$$W_0[\,J\,] = \frac{i}{2}\int dx\int dy\;\, J(x)\,D_F\,(x-y)\,J(y) \equiv \frac{i}{2}\,\langle\, J_x\,D_{xy}\,J_y\,\rangle$$

and define the so called *classical field* $\phi_{c\ell}\,(x)$ by the relation

$$\phi_{c\ell}\,(x) \;\stackrel{\mathrm{def}}{=}\; \delta\,W_0\,[\,J\,]\,/\,\delta\,J(x) = i\int dy\;\, D_F\,(x-y)\,J(y) \tag{4.55}$$

Thus we immediately find

$$\left(\Box + m^2\right)\phi_{c\ell}\,(x) = J(x) \tag{4.56}$$

that is the Klein−Gordon free wave equation in the presence of a classical external source $J(x)$. Let us perform the functional Legendre transformation

$$\begin{aligned}\Gamma_0[\,\phi_{c\ell}\,] &\stackrel{\text{def}}{=} W_0[\,J\,]-\int \mathrm{d}x\ \phi_{c\ell}(x)\,J(x)\\ &= W_0[\,J\,]-\int \mathrm{d}x\ \phi_{c\ell}(x)\,\left(\Box+m^2\right)\phi_{c\ell}(x)\\ &= -\int \mathrm{d}x\ \phi_{c\ell}(x)\,\tfrac{1}{2}\left(\Box+m^2\right)\phi_{c\ell}(x)\\ &\doteq \int \mathrm{d}x\ \tfrac{1}{2}\left[\,\partial_\mu\,\phi_{c\ell}(x)\,\partial^\mu\,\phi_{c\ell}(x)-m^2\,\phi_{c\ell}^2(x)\,\right]\end{aligned} \tag{4.57}$$

which is nothing but the classical action for a free real scalar field. A similar procedure can be closely carried out in the case of a self−interacting real scalar field with *e.g.* quartic potential $V[\,\phi\,]=(-\,\lambda/4!)\int dx\ \phi^4(x)$. Let me first define the functional Legendre transformation in the interacting case

$$\phi_{c\ell}(x)\ \stackrel{\text{def}}{=}\ \frac{\delta W[\,J\,]}{\delta\,J(x)} \tag{4.58}$$

$$\Gamma[\,\phi_{c\ell}\,]\ \stackrel{\text{def}}{=}\ W[\,J\,]-\int \mathrm{d}x\ \phi_{c\ell}(x)\,J(x)\equiv W[\,J\,]-\langle\,\phi_{c\ell}\,J\,\rangle \tag{4.59}$$

$$J(x)\ \equiv\ -\,\frac{\delta\,\Gamma[\,\phi_{c\ell}\,]}{\delta\,\phi_{c\ell}(x)} \tag{4.60}$$

as it is clear from (4.59) because $\Gamma\,[\,\phi_{c\ell}\,]$ depends solely upon the classical field $\phi_{c\ell}$ while the generating functional $W\,[\,J\,]$ of the connected Green's functions depends only on the external sources J . From the expression

$$\begin{aligned} Z[\,J\,] &= \exp\{\,iW[\,J\,]\}\\ &= \exp\{-\,\mathrm{i}V[\,\delta/i\,\delta\,J]\}\ Z_0[\,J\,]\\ &= \exp\{-\,\mathrm{i}V[\,\delta/i\,\delta\,J]\}\ \exp\left\{-\,\tfrac{1}{2}\,\langle\,J_x\,D_{xy}\,J_y\,\rangle\right\}\end{aligned} \tag{4.61}$$

we obtain

$$\begin{aligned}\frac{\delta\,Z}{\delta\,J_x} &= -\ \exp\{-\,\mathrm{i}V[\,\delta/i\,\delta\,J]\}\ \langle\,D_{xy}\,J_y\,\rangle\,Z_0[\,J\,]\\ &= -\ \exp\{-\,\mathrm{i}V[\,\delta/i\,\delta\,J]\}\ \langle\,D_{xy}\,J_y\,\rangle\ \exp\{iV[\,\delta/i\,\delta\,J]\}\ Z[\,J\,]\end{aligned}$$

Hence it follows that we can write the functional equation

$$\left(\Box_x+m^2\right)\frac{\delta\,Z}{\delta\,J_x}=i\,O_x\,Z[\,J\,] \tag{4.62}$$

in which I have set

$$
\begin{aligned}
O_x &\stackrel{\text{def}}{=} \exp\{-\mathrm{i}V[\delta/i\,\delta J]\}\ J_x\ \exp\{iV[\delta/i\,\delta J]\} \\
&= J_x - V'[\delta/i\,\delta J(x)]
\end{aligned}
\tag{4.63}
$$

where V' is the derivative of V with respect to its argument.

`Proof.`

Let me set

$$
O_x(a) = \exp\{-\mathrm{i}aV[\delta/i\,\delta J]\}\ J_x\ \exp\{iaV[\delta/i\,\delta J]\}
$$

where a is a real parameter. Differentiating we find

$$
\frac{\mathrm{d}}{\mathrm{d}a}\,O_x(a) = \exp\{-\mathrm{i}aV[\delta/i\,\delta J]\}\ [-\mathrm{i}V[\delta/i\,\delta J]\,,\,J_x]\ \exp\{iaV[\delta/i\,\delta J]\}
$$

On the other side we get

$$
\begin{aligned}
[-\mathrm{i}V[\delta/i\,\delta J]\,,\,J_x] &= \int \mathrm{d}y\ \ [-\mathrm{i}V[\delta/i\,\delta J_y]\,,\,J_x] = \\
-\int dy\ V'[\delta/i\,\delta J_y]\,\delta(x-y) &= -\,V'[\delta/i\,\delta J_x]
\end{aligned}
\tag{4.64}
$$

and thereby

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a}\,O_x(a) + V'[\delta/i\,\delta J_x] &= 0 \\
\int_0^1 \mathrm{d}a\ \frac{d}{da}\,O_x(a) = O_x - J_x &= -\,V'[\delta/i\,\delta J_x] \\
O_x \ = \ J_x \ - \ V'[\delta/i\,\delta J_x]
\end{aligned}
\tag{4.65}
$$

Quod Erat Demonstrandum

Hence

$$
\begin{aligned}
\left(\Box_x + m^2\right)\,\frac{\delta Z}{\delta J_x} &= (i\,J_x\ -\ iV'[\delta/i\,\delta J_x])\ Z[J] \\
\left(\Box_x + m^2\right)\,\frac{\delta W}{\delta J_x} &= J_x\ -\ \frac{1}{Z[J]}\,V'[\delta/i\,\delta J_x]\,Z[J] \\
\left(\Box_x + m^2\right)\,\phi_{c\ell}(x) &= J_x\ -\ \mathrm{e}^{-\mathrm{i}W[J]}\,V'[\delta/i\,\delta J_x]\,Z[J]
\end{aligned}
\tag{4.66}
$$

the very last term just looking like a kind of driving functional force. Now we have

$$
\begin{aligned}
\mathrm{e}^{-\mathrm{i}W[J]}\,V'[\delta/i\,\delta J_x]\,\mathrm{e}^{\mathrm{i}W[J]} &= \frac{\lambda}{3!}\,(-\mathrm{i})^3\,\mathrm{e}^{-\mathrm{i}W[J]}\,\frac{\delta^3}{\delta J_x^3}\,\mathrm{e}^{\mathrm{i}W[J]} \\
&= \frac{\lambda}{3!}\left[\phi_{c\ell}^2(x) - \frac{\delta^2}{\delta J_x^2} - 3\,i\,\phi_{c\ell}(x)\,\frac{\delta}{\delta J_x}\right]\phi_{c\ell}(x)
\end{aligned}
\tag{4.67}
$$

and finally

$$\left(\Box_x + m^2\right)\,\phi_{c\ell}(x) = J_x\; - \frac{\lambda}{3!}\left[\phi^3_{c\ell}(x) - \frac{\delta^2\phi_{c\ell}}{\delta J_x^2} - 3\,i\,\phi_{c\ell}(x)\,\frac{\delta\phi_{c\ell}}{\delta J_x}\right] \quad (4.68)$$

The effective action in the interacting case can not be written in closed form and turns out to be a non−local functional of the classical field function $\phi_{c\ell}(x)$. We can write

$$\Gamma[\,\phi_{c\ell}\,] = \sum_{n=0}^{\infty}\frac{1}{n!}\int \mathrm{d}x_1\;\phi_{c\ell}(x_1)\;\ldots\;\int \mathrm{d}x_n\;\phi_{c\ell}(x_n)\;\Gamma^{(n)}(\,x_1\,,\,\ldots\,,\,x_n\,) \quad (4.69)$$

where the non−local coefficients $\Gamma^{(n)}(\,x_1\,,\,\ldots\,,\,x_n\,)$ are named the $n-$point *proper vertices* or *strongly connected Green's functions* and turns out to be translation invariant. Thus their Fourier transforms read

$$\begin{aligned}\widetilde{\Gamma}^{(n)}(\,k_1\,,\,\ldots\,,\,k_n\,)\,(2\pi)^4\,\delta\,(\,k_1 + \,\cdots\, + k_n\,) \;=\\ \int \mathrm{d}x_1\;\ldots\;\int \mathrm{d}x_n\;\mathrm{e}^{\mathrm{i}k_1x_1+\cdots+\mathrm{i}k_nx_n}\;\Gamma^{(n)}(\,x_1\,,\,\ldots\,,\,x_n\,)\end{aligned} \quad (4.70)$$

As a consequence we can eventually write

$$\Gamma^{(n)}(\,x_1\,,\,\ldots\,,\,x_n\,) = \delta^{(n)}\,\Gamma[\,\phi_{c\ell}\,]/\,\delta\,\phi_{c\ell}(x_1)\;\ldots\;\delta\,\phi_{c\ell}(x_n)\big\rfloor_{\,\phi_{c\ell}=0} \quad (4.71)$$

and in particular

$$\begin{aligned}\Gamma^{(2)}(\,x-y\,) \;&=\; \delta^{(2)}\,\Gamma[\,\phi_{c\ell}\,]/\,\delta\,\phi_{c\ell}(x)\,\delta\,\phi_{c\ell}(y)\big\rfloor_{\,\phi_{c\ell}=0}\\ &=\; \int\frac{\mathrm{d}^4k}{(2\pi)^4}\;\widetilde{\Gamma}^{(2)}(\,k\,)\;\mathrm{e}^{-\,\mathrm{i}\,k\,x}\end{aligned} \quad (4.72)$$

Notice that the functional derivative of the inverse functional leads to the following remarkable relation : namely,

$$\begin{aligned}\delta^{(2)}\,W[\,J\,]/\,\delta J(y)\,\delta J(x) \;&=\; \delta\phi_{c\ell}(x)/\,\delta J(y)\\ =\; \left[\delta J(y)/\,\delta\phi_{c\ell}(x)\right]^{-1} \;&=\; -\,\left[\delta^{(2)}\,\Gamma/\,\delta\phi_{c\ell}(x)\,\delta\phi_{c\ell}(y)\right]^{-1}\end{aligned} \quad (4.73)$$

that means

$$\int \mathrm{d}y\;\frac{\delta^{(2)}\,W[\,J\,]}{\delta J(x)\,\delta J(y)}\cdot\frac{\delta^{(2)}\,\Gamma[\,\phi_{c\ell}\,]}{\delta\phi_{c\ell}(y)\,\delta\phi_{c\ell}(z)} \;=\; -\,\delta\,(x-z) \quad (4.74)$$

and after setting the external sources and the classical fields equal to zero

$$\int \mathrm{d}y\;G^{(2)}(\,x-y\,)\,\Gamma^{(2)}(\,y-z\,) \;=\; i\,\delta\,(x-z) \quad (4.75)$$

or in momentum space

$$\widetilde{G}^{(2)}(k)\,\widetilde{\Gamma}^{(2)}(k) \;=\; i \tag{4.76}$$

In the non−interacting case the above relation reduces to the trivial identity

$$\widetilde{G}_0^{(2)}(k)\,\widetilde{\Gamma}_0^{(2)}(k) \;\equiv\; \frac{i}{k^2 - m^2 + \mathrm{i}\varepsilon} \;\cdot\; (k^2 - m^2) \;=\; i$$

The Fourier transform $\widetilde{G}^{(2)}(k)$ of the 2−point function $G^{(2)}(x-y)$ is customarily named the *full or exact propagator*

$$G^{(2)}(x-y) = \langle\, 0\,|\, T\,\phi(x)\,\phi(y)\,|\,0\,\rangle = \int \frac{\mathrm{d}k}{(2\pi)^4}\,\widetilde{G}^{(2)}(k)\,\mathrm{e}^{-\mathrm{i}\,k\,x} \tag{4.77}$$

In the interacting case it is customary [6] to introduce the *scalar self−energy* invariant function by means of the relationships

$$\widetilde{G}^{(2)}(k) \stackrel{\text{def}}{=} \frac{i}{k^2 - m^2 - \Sigma(k) + \mathrm{i}\varepsilon} \tag{4.78}$$

$$\widetilde{\Gamma}^{(2)}(k) = k^2 - m^2 - \Sigma(k) \tag{4.79}$$

It is important to realize that we can write the famous Schwinger−Dyson equation for the full or exact scalar propagator, *i.e.* ,

$$\begin{aligned}
\widetilde{G}^{(2)}(k) &\stackrel{\text{def}}{=} \widetilde{G}_0^{(2)}(k) + \widetilde{G}_0^{(2)}(k)\,\frac{1}{\mathrm{i}}\,\Sigma(k)\,\widetilde{G}_0^{(2)}(k) + \cdots \\
&= \widetilde{G}_0^{(2)}(k)\sum_{n=0}^{\infty}\left[\frac{1}{\mathrm{i}}\,\Sigma(k)\,\widetilde{G}_0^{(2)}(k)\right]^{n} \\
&= \mathrm{i}\left[\widetilde{\Gamma}_0^{(2)}(k) - \Sigma(k)\right]^{-1} \;=\; \left[\frac{1}{\mathrm{i}}\,\widetilde{\Gamma}^{(2)}(k)\right]^{-1}
\end{aligned} \tag{4.80}$$

It turns out that the scalar self−energy invariant function $\Pi(k)$ does correspond by construction to the sum of all the 1PI 2−point diagrams amputated by their two external free propagators, *i.e.* the sum of all the 2−point proper vertices (the *sausage* sum)

$$\widetilde{G}^{(2)}(k) = \widetilde{G}_0^{(2)}(k) - \mathrm{i}\,\widetilde{G}_0^{(2)}(k)\,\Sigma(k)\,\widetilde{G}^{(2)}(k) \tag{4.81}$$

In a quite analogous way we define the spinor self−energy matrix

$$\widetilde{G}^{(2)}(p) \stackrel{\text{def}}{=} \frac{i}{\not{p} - M - \Sigma(\not{p}) + \mathrm{i}\varepsilon} \tag{4.82}$$

$$\widetilde{\Gamma}^{(2)}(p) = \not{p} - M - \Sigma(\not{p}) \tag{4.83}$$

[6] Here I follow the convention and notation of [12] §**6-2-2** eq. (6-78) p. 291.

In the photon field case, the exact photon propagator, *i.e.* , the exact 2−point Green's function, is defined by

$$\begin{aligned}\widetilde{G}^{\mu\nu}(k) &\stackrel{\text{def}}{=} \widetilde{G}_0^{\mu\nu}(k) + \widetilde{G}_0^{\mu\rho}(k)\left[\mathrm{i}\left(k^2 g_{\rho\sigma} - k_\rho k_\sigma\right)\Pi(k^2)\right]\widetilde{G}_0^{\sigma\nu}(k)\\ &+ \cdots\\ &= \widetilde{G}_0^{\mu\nu}(k) + \widetilde{G}_0^{\mu\rho}(k)\,P_\rho^{\ \nu}\,\Pi(k^2) + \widetilde{G}_0^{\mu\rho}(k)\,P_\rho^{\ \sigma}\,P_\sigma^{\ \nu}\,\Pi^2(k^2)\\ &+ \cdots \end{aligned} \tag{4.84}$$

where I have introduced the off−shell transverse projector

$$P_\rho^{\ \nu} \stackrel{\text{def}}{=} \delta_\rho^{\ \nu} - k_\rho k^\nu / k^2 \qquad (k^2 \neq 0) \tag{4.85}$$

which satisfy $P_\rho^{\ \sigma} P_\sigma^{\ \nu} = P_\rho^{\ \nu}$. Then we can simplify the above expression to

$$\begin{aligned}\widetilde{G}^{\mu\nu}(k) &= \widetilde{G}_0^{\mu\nu}(k) + \widetilde{G}_0^{\mu\rho}(k)\,P_\rho^{\ \nu}\left[\Pi(k^2) + \Pi^2(k^2) + \cdots\right]\\ &= \frac{-\mathrm{i}}{k^2[1-\Pi(k^2)]}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \frac{-\mathrm{i}}{k^2}\left(\frac{k^\mu k^\nu}{k^2}\right)\end{aligned} \tag{4.86}$$

Thus, in the case of the photon field, we are eventually led to the gauge invariant polarization function $\Pi(k^2)$, so that we can identify the 2−point transverse gauge invariant proper vertex with

$$\widetilde{\Gamma}_{\mu\nu}(k^2) \stackrel{\text{def}}{=} (-k^2 g_{\mu\nu} + k_\mu k_\nu)\,[1 - \Pi(k^2)] \tag{4.87}$$

and consequently

$$\widetilde{G}^{\mu\rho}(k)\,\widetilde{\Gamma}_{\rho\nu}(k^2) = \mathrm{i}\,\delta^\mu_\nu \tag{4.88}$$

If we pass to the 3−point Green's functions, from the main relation (4.74) taking one more functional derivative with respect to the external sources we can write

$$\begin{aligned}&\int \mathrm{d}y\;\frac{\delta^{(3)} W[J]}{\delta J(x)\,\delta J(y)\,\delta J(\xi)}\cdot\frac{\delta^{(2)}\Gamma[\phi_{c\ell}]}{\delta\phi_{c\ell}(y)\,\delta\phi_{c\ell}(z)} +\\ &\int \mathrm{d}y\;\frac{\delta^{(2)} W[J]}{\delta J(x)\,\delta J(y)}\cdot\frac{\delta^{(3)}\Gamma[\phi_{c\ell}]}{\delta\phi_{c\ell}(y)\,\delta\phi_{c\ell}(z)\,\delta J(\xi)} = 0\end{aligned} \tag{4.89}$$

and using the functional relation

$$\begin{aligned}\frac{\delta}{\delta J(\xi)}\left(\frac{\delta^{(2)}\Gamma[\phi_{c\ell}]}{\delta\phi_{c\ell}(x)\,\delta\phi_{c\ell}(y)}\right) &= \int \mathrm{d}w\;\frac{\delta^{(3)}\Gamma[\phi_{c\ell}]}{\delta\phi_{c\ell}(x)\,\delta\phi_{c\ell}(y)\,\delta\phi_{c\ell}(w)}\cdot\frac{\delta\phi_{c\ell}(w)}{\delta J(\xi)}\\ &= \int \mathrm{d}w\;\frac{\delta^{(3)}\Gamma[\phi_{c\ell}]}{\delta\phi_{c\ell}(x)\,\delta\phi_{c\ell}(y)\,\delta\phi_{c\ell}(w)}\cdot\frac{\delta^{(2)} W[J]}{\delta J(w)\,\delta J(\xi)}\end{aligned}$$

so that, after putting eventually the external sources and the classical fields equal to zero, we get

$$\frac{\delta^{(3)} W}{\delta J_x\, \delta J_y\, \delta J_\xi} \cdot \frac{\delta^{(2)} \Gamma}{\delta \varphi_y\, \delta \varphi_z} + \frac{\delta^{(2)} W}{\delta J_x\, \delta J_y} \cdot \frac{\delta^{(2)} W}{\delta J_\xi\, \delta J_w} \cdot \frac{\delta^{(3)} \Gamma}{\delta \varphi_w\, \delta \varphi_y\, \delta \varphi_z} = 0$$

where I have suitably introduced a discrete−like index notation, the sum over repeated indices being understood, together with $\phi_{c\ell} \equiv \varphi$. Hence, if we remember that

$$\left. \frac{\delta^{(n)} W}{\delta J_1 \, \ldots \, \delta J_n} \right]_{J=0} = i^{\,n-1}\, G_c^{(n)}(x_1\,,\,\ldots\,,\,x_n) \qquad G_{xy}^{(2)}\,\Gamma_{yz}^{(2)} = \mathrm{i}\,\delta_{xz}$$

$$\left. \frac{\delta^{(n)} \Gamma}{\delta \varphi_1 \, \ldots \, \delta \varphi_n} \right]_{\varphi=0} = \Gamma^{(n)}(x_1\,,\,\ldots\,,\,x_n)$$

it readily follows

$$G_{xy\xi}^{(3)}\,\Gamma_{yz}^{(2)} = -\,G_{xy}^{(2)}\,G_{\xi w}^{(2)}\,\Gamma_{wyz}^{(3)} \tag{4.90}$$

and thereby

$$\Gamma_{vx}^{(2)}\,\Gamma_{u\xi}^{(2)}\,G_{xy\xi}^{(3)}\,\Gamma_{yz}^{(2)} = \Gamma_{uvz}^{(3)} \tag{4.91}$$

This equality allows us to identify $\Gamma_{uvz}^{(3)}$ with the 3−point proper vertex

$$\begin{aligned} \left. \frac{\delta^{(3)} \Gamma}{\delta \varphi_w\, \delta \varphi_y\, \delta \varphi_z} \right]_{\varphi=0} &= \Gamma^{(3)}(w\,,\,y\,,\,z) \\ &= \delta^{(3)}\Gamma[\,\phi_{c\ell}\,]/\delta\,\phi_{c\ell}(w)\,\delta\,\phi_{c\ell}(y)\,\delta\,\phi_{c\ell}(z)\big\rfloor_{\,\phi_{c\ell}=0} \end{aligned} \tag{4.92}$$

Actually, the 3−point proper vertex $\Gamma^{(3)}(w, y, z)$ is nothing but the connected 3−point Green's function in which the external complete propagators have been amputated, *i.e.* the 1PI strongly connected 3−point Green's function. By iterating the above described procedure, it can be shown by induction that the effective action is the generating functional of all the n−point proper vertices.

In going to the euclidean formulation, it is convenient to define

$$Z_E = \mathrm{e}^{-\,W_E} \qquad \varphi_E(\bar{x}) = -\,\frac{\delta W_E[\,J_E\,]}{\delta J_E(x_E)}$$

$$\Gamma_E[\,\varphi_E\,] = \langle\, J_E\,\varphi_E\,\rangle + W_E[\,J_E\,] = \int \mathrm{d}\bar{x}\;\; J_E(\bar{x})\,\varphi_E(\bar{x}) + W_E[\,J_E\,]$$

$$J_E(\bar{x}) = \frac{\delta\,\Gamma_E[\,\varphi_E\,]}{\delta\,\varphi_E(\bar{x})}$$

so that the euclidean effective action is nothing but the Gibbs free enthalpy in natural units $kT = 1 = 1/\beta$, where k is the Boltzmnn constant and T the absolute temperature. As a matter of fact, we nicely eventually come to the following correspondences among euclidean functionals and statistical mechanics entities in natural units : namely,

Z_E	canonical partition function (*Zustandsumme*)
W_E	Helmoltz free energy
J_E	generalized external parameter (*volume*)
φ_E	generalized external force (*pressure*)
Γ_E	Gibbs free enthalpy

We can directly obtain

$$\frac{\delta\,\varphi_E(\bar{x})}{\delta J_E(\bar{y})} \;=\; -\,\frac{\delta^{(2)}\,W_E[\,J_E\,]}{\delta J_E(\bar{x})\,\delta J_E(\bar{y})} \;=\; \left[\,\frac{\delta^{(2)}\,\Gamma_E[\,\varphi_E\,]}{\delta\,\varphi_E(\bar{x})\,\delta\,\varphi_E(\bar{y})}\,\right]^{-1} \tag{4.93}$$

and taking into account the definition (1.49) we come to the simple relation

$$\int \mathrm{d}\bar{y}\; G_E^{(2)}(\bar{x}-\bar{y})\,\Gamma_E^{(2)}(\bar{y}-\bar{z}) \;=\; \delta(\bar{x}-\bar{z}) \tag{4.94}$$

$$\widetilde{G}_E^{(2)}(\bar{k})\,\widetilde{\Gamma}_E^{(2)}(\bar{k}) \;=\; 1 \tag{4.95}$$

where I have set

$$\Gamma_E^{(n)}(\bar{x}_1\,,\,\ldots\,,\,\bar{x}_n) \;=\; \delta^{(n)}\,\Gamma_E[\,\varphi_E\,]\,/\,\delta\,\varphi_E(\bar{x}_1)\,\ldots\,\delta\,\varphi_E(\bar{x}_n)\,\Big]_{\varphi_E=0} \tag{4.96}$$

4.2 Divergences of Feynman diagrams

4.2.1 Power counting criterion

Consider a Feynman diagram with V vertices, E external lines, *i.e.* carrying incoming or outgoing external momenta, and I internal lines. To warm up, let me assume for a start that only scalar particles are involved. The number of independent internal momenta is the number of loops L of the diagram. The I internal momenta do satisfy $V-1$ relations among themselves, the -1 appearing just owing to overall energy−momentum conservation, so that

$$L = I - V + 1 \tag{4.97}$$

This equality allows us to compute the naïve power counting of momenta for the diagram which provides the *superficial degree of divergence* $\omega\,(G)$ of the Feynman graph G – here superficial stands for apparent. To determine $\omega\,(G)$ we note that there are

- L independent loop integrations, each one providing D powers of the momenta in $D-$dimensions
- I internal momenta, each one providing a scalar Feynman propagator with two inverse powers of the momenta. Hence

$$\omega\,(G) = DL - 2I$$

We need one more relation among V, E and I. Let me denote by V_N the number of vertices with N legs, *i.e.* N concurring entering momenta. In a graph G with V_N such vertices we have NV_N lines which are either internal or external. It turns out that any internal line counts twice, for it originates and terminates at some vertex, in such a manner that

$$NV_N = E + 2I$$

The above relationships allows us to express the superficial degree of divergence in terms of the number of external lines, the number of vertices and the number of space−time dimensions

$$\omega\,(G) = D - \frac{1}{2}\,(\,D-2\,)\,E + V_N\left(\frac{N-2}{2}\,D - N\right) \tag{4.98}$$

In four dimensions $D = 4$ we find

$$\omega\,(G) = 4 - E + (\,N-4\,)\,V_N \qquad [\,\text{four dimensions}\,]$$

and in the case of the quartic scalar self−interaction

$$\omega\,(G) = 4 - E \qquad\qquad [\,\lambda\,\phi^4 \text{ in } D = 4\,]$$

The key result here is that the superficial degree of divergence $\omega\,(G)$ does not depend upon the number of vertices but solely on the number of external legs. Thus we have only two candidates with $\omega\,(G) \geq 0$

- $\widetilde{G}_c^{(2)}\,(k)$ with $\omega\,(G) = 2$ (quadratic divergence)
- $\widetilde{G}_c^{(4)}\,(k_1, k_2, k_3, k_4)$ with $\omega\,(G) = 0$ (logarithmic divergence)

Note that these 2− and 4−point Green's functions are directly related to the kinetic and interaction terms of the classical Lagrange density, a feature that we be proved to be crucial for a successfull renormalization program to all orders in perturbation theory.

For example we have already seen in equation (4.7) that the lowest order contribution to $\widetilde{G}_2\,(k)$ is given by

$$\begin{aligned}\widetilde{G}^{(2)}\,(k) \;&=\; \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon} \\ &\times\; \left\{1 + \frac{\mathrm{i}\lambda}{16\pi^2}\left(K^2 - m^2\left[\ln\frac{K}{m} - \frac{1}{2} + \ln 2 + O\left(\frac{m}{K}\right)^2\right]\right)\right. \\ &\times\; \left.\frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon}\right\} \;+\; O\,(\,\lambda^2\,) \qquad (4.99)\end{aligned}$$

where $K \sim M_{\mathrm{P}}$ is a large ultraviolet cut−off. On the other side, using dimensional regularisation, the very same Green's function reads

$$\begin{aligned}\widetilde{G}^{(2)}\,(k) \;&=\; \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon} \\ &\times\; \left\{1 + \frac{\mathrm{i}\lambda m^2}{32\,\pi^2}\left[\frac{1}{\epsilon} + \psi\,(2) - \ln\frac{m^2}{4\pi\mu^2}\right]\frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon}\right\} \\ &+\; O\,(\,\lambda^2\,) \qquad (4.100)\end{aligned}$$

while equation (4.27) yields

$$\begin{aligned}\widetilde{G}^{(4)}\,(k_1, k_2, k_3, k_4) \;&=\; (-\,\mathrm{i}\lambda)\prod_{\jmath=1}^{4}\frac{\mathrm{i}}{k_\jmath^2 - m^2 + \mathrm{i}\varepsilon} \;\times \\ \left\{1 - \frac{3\lambda}{32\pi^2}\left[\frac{1}{\epsilon} - \mathbf{C} + 2 + \ln\frac{4\pi\mu^2}{m^2} - \frac{1}{3}\,A\,(\,s, t, u\,)\right] + O\,(\,\lambda^2\,)\right\}& \qquad (4.101)\end{aligned}$$

with $2\epsilon = 4 - D$ and for $0 < z < 4m^2$

$$A(s,t,u) = \sum_{z=s,t,u} \left(\frac{4m^2}{z} - 1 \right)^{1/2} \operatorname{arcctg} \sqrt{\frac{4m^2}{z} - 1}$$

where s, t and u are the Mandelstam' variables

$$s = (k_1 + k_2)^2 \qquad t = (k_1 + k_3)^2 \qquad u = (k_1 + k_4)^2$$

The above analysis does not prove that $\widetilde{G}_c^{(n)}(k_1, \ldots k_n) \quad (n \in \mathbb{N}, n > 4)$ which exhibit a negative superficial degree of divergence $\omega(G) < 0$ are finite in four dimensions, because they contain sub−divergences. To this concern, the necessary and sufficient condition for the convergence of a Feynman graph G is provided by the so called Weinberg's theorem [7]

Steven Weinberg,
High−Energy Behavior in Quantum Field Theory,
Physical Review 118, 838 - 849 (1960) [Issue 3 – May 1960]

In order to state this convergence theorem, let me first first recall that the Green's functions and corresponding Feynman diagrams can be divided into disconnected and connected ones, the latter being just characterized by the property that all vertices of the corresponding Feynman graph are connected by at least one internal line. In general, a connected graph G is 1−particle reducible (1PR) in the sense that it can be separated into two disconnected subgraphs by cutting an internal line. Conversely, we shall call *strongly connected* or 1−particle irreducible (1PI) any Feynman graph G that can not be separated into two disconnected subgraphs by cutting one of its internal lines. Finally, we shall call *proper vertices* the 1PI Green's functions in which all its external lines have been amputated. The 1PI or strongly connected Green's functions or even n−point proper vertices in momentum space are commonly denoted by $\widetilde{\Gamma}^{(n)}(k_1, \ldots, k_n)$, which precisely correspond to the momentum space expansion coefficients (4.70) of the effective action.

I will use here a restrictive definition of a sub−diagram $g \subset G$ of a diagram G : this is a subset of vertices of G and of all internal lines joining them in G. Then, to each strongly connected, *i.e.* 1−particle irreducible, graph G we associate the set $\mathcal{F}$ of all its strongly connected sub−graphs $g \in G$. Of course, $\mathcal{F}$ contains G itself.

[7] See *e.g.* N.N. Bogolyubov and D.M. Shirkov, *Introduction to the Theory of Quantized Fields*, Interscience Publishers, New York, 1959 ; C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill, New York, 1980, §**8–1–4** pp. 382 – 385.

Theorem If $\omega(g) < 0\,,\ \forall\, g \in \mathcal{F}$ then the Feynman integral which corresponds to G is absolutely convergent in the euclidean formulation.

This means that in the $\lambda\,\phi^4$ perturbative quantum field theory in four space-time dimensions, the generic sources of divergences are the two−point and the four−point proper vertices and nothing else. Thus, if we will be able to remove the divergences, order by order in perturbation theory, from $\widetilde{\Gamma}^{(2)}(k_1)$ and $\widetilde{\Gamma}^{(4)}(k_1\,,\,\ldots\,,\,k_4)$, then all other Green's functions $\widetilde{G}^{(n)}(k_1\,,\,\ldots\,,\,k_n)$ of the theory will be divergence free. To the lowest order we have

$$\begin{aligned}\widetilde{\Gamma}^{(2)}(k) &= \mathrm{i}\,[\,\widetilde{G}^{(2)}(k)\,]^{-1} \qquad (4.102)\\ &= k^2 - m^2\left\{1-\frac{\lambda}{32\pi^2}\left[\frac{1}{\epsilon}+\psi(2)-\ln\frac{m^2}{4\pi\mu^2}\right]\right\}+O(\lambda^2)\end{aligned}$$

which corresponds, up to an irrelevant four−divergence, to the kinetic term of the classical Lagrange density

$$\begin{aligned}&\tfrac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi(x)\,\partial_\nu\phi(x)-\tfrac{1}{2}\,m^2\,\phi^2(x)\\ \times\ &\left\{1-\frac{\lambda}{32\pi^2}\left[\frac{1}{\epsilon}+\psi(2)-\ln\frac{m^2}{4\pi\mu^2}\right]\right\}\ +\ O(\lambda^2)\end{aligned} \qquad (4.103)$$

Furthermore

$$\begin{aligned}\widetilde{\Gamma}^{(4)}(k_1,k_2,k_3,k_4) &= (-\,\mathrm{i}\lambda)\\ &\times\ \left\{1-\frac{3\lambda}{32\pi^2}\left[\frac{1}{\epsilon}-\mathbf{C}+2+\ln\frac{4\pi\mu^2}{m^2}-\frac{1}{3}\,A(s,t,u)\right]\right\}\\ &+\ O(\lambda^3)\end{aligned} \qquad (4.104)$$

the divergent part of which does correspond to the interaction potential

$$\frac{\lambda}{4!}\,\phi^4(x)\left(1-\frac{3\lambda}{32\pi^2}\cdot\frac{1}{\epsilon}\right)+O(\lambda^2)$$

The graphs which contain the generic superficial divergences are named to be *primitively divergent*. The fact that in the quartic self−interacting real scalar field theory the primitively divergent graphs are finite in species (two− and four−point proper vertices) and correspond precisely to the type of monomials appearing in the classical Lagrangian density, is the *necessary condition* for the successfull removal of all the ultraviolet divergences to all orders in perturbation theory. A field theory model for which this is possible is said *power counting renormalizable*. Actually, we can easily realize from the expression (4.98) for the superficial degree of divergence that quite a few number of scalar theories does fulfill this key requirement.

- In four space−time dimensions, *i.e.* $D = 4$, we see that $\omega(G)$ grows with the number of vertices V_N with $N > 4$. This means that the scalar self−interactions of higher powers $g_N\,\phi^N(x)$ $(N > 4)$, although perfectly reasonable classically, necessarily lead at the quantum level to an infinite number of primitively divergent graphs. In such a nasty case, the situation gets quickly out of control and the possibility to remove the divergences to all orders in perturbation theory indeed disappears and thereby renormalizability is lost.

- In $D = 4$ and with $N > 4$ the coupling g_N have the canonical mass dimensions $[\,g_N\,] = \mathrm{eV}^{\,4-N}$. This fact strongly suggests that the very criterion of power counting renormalizability is deeply connected, in four space−time dimensions, with a dimensionless coupling parameter for the interaction. This simple observation immediately led Werner Heisenberg to realize that the quantization of a field theory such as Einstein' general relativity appears to be a formidable task. In fact, the Newton's constant G is nothing but, in natural units, the square of the Planck length $G = \ell_{\mathrm{P}}^2\,c^3/\hbar$ or the inverse square of the Planck mass $G = \hbar c/M_{\mathrm{P}}^2$. It follows that any probability amplitude involving quantized gravity will exhibit the $n-$th order radiative correction

$$\mathcal{A}^{(n)} \;\propto\; G^n \int_0^\Lambda \ell^{2n-1}\,d\ell \;\sim\; G^n\,\Lambda^{2n}$$

  where Λ is the ultraviolet cut−off, which evidently implies an infinite variety of infinities, their species being increasing *ad libitum* with the order of the perturbative expansion [8]. This means that the quantization of Einstein' general relativity can not give rise to a perturbatively power conting renormalizable quantum field theory.

- When $D = 2$, *i.e.* one space and one time dimensions, the situation is completely reversed. There we have

$$\omega(G) = 2 - 2V_N \qquad (\text{two space} - \text{time dimensions})$$

  in such a manner that the superficial degree of divergence does not depend upon N which labels the type of interaction. It depends only on the number of vertices and the more vertices are there the more convergent is the Feynman integral. Moreover, the degree $\omega(G)$ is

[8] This simple dimensional argument is reported by Steven Weinberg, *Gravitation and cosmology : principles and applications of the general theory of relativity*, John Wiley & Sons, New York, 1972, chapter X § **8** p. 289.

independent of the number E of the external legs, so that the only primitively divergent diagrams have one or zero legs. Since divergences occur owing to loop integrals, this means that the latter occur only when a propagator from a vertex is closed on the very same vertex. But this is precisely the tadpole graph, the divergence of which can be removed by the normal ordering prescription. In other words, if we start from the free field quantized action

$$S_0 = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \ \tfrac{1}{2} : \partial_\mu \phi(t,x)\, \partial^\mu \phi(t,x) - m^2 \phi^2(t,x) :$$

then no divergences will appear in perturbation theory for the Green's functions. Incidentally, this is the ultimate simple reason why the method of dimensional regularization in the euclidean formulation does render all the Feynman integral absolutely convergent.

Now we are ready to argue about the interacting field theories involving spinor fields, such as the Yukawa meson theory or quantum electrodynamics. Once again, it turns out that the number of possible fermion interactions is drastically reduced by the very strong requirement of the power counting renormalizability, which demands as a necessary condition that the number of species of primitively divergent graphs were finite. Let us therefore compute the superficial degree of divergence $\omega(G)$ on an arbitrary Feynman diagram involving scalar, spinor and vector fields.

Consider therefore a generic Feynman graph G with L loops, I_{b} boson internal lines, I_{f} fermion internal lines, V vertices with N_{b} boson and N_{f} fermion concurring lines, E_{b} external boson lines and E_{f} external fermion lines. As already repeatedly remarked the numbers N_{f} and E_{f} must be even. The number of loops is given by

$$L = I - V + 1 = I_{\mathrm{b}} + I_{\mathrm{f}} - V + 1$$

The superficial degree of divergence in D space−time dimensions is

$$\omega(G) = D \cdot L - I_{\mathrm{f}} - 2\, I_{\mathrm{b}}$$

since each spinor propagator contributes only one power of momentum. In addition, the total number of fermionic spinor lines is given by

$$V N_{\mathrm{f}} = E_{\mathrm{f}} + 2 I_{\mathrm{f}}$$

and similarly for bosonic scalar and vector lines

$$V N_{\mathrm{b}} = E_{\mathrm{b}} + 2 I_{\mathrm{b}}$$

The above relations enable us to express the superficial degree of divergence in the form

$$\begin{aligned}\omega\,(G) &= D - \frac{1}{2}\,(\,D-1\,)\,E_{\rm f} - \frac{1}{2}\,(\,D-2\,)\,E_{\rm b} \\ &- V\left[\,D - \frac{1}{2}\,(\,D-1\,)\,N_{\rm f} - \frac{1}{2}\,(\,D-2\,)\,N_{\rm b}\right] \qquad (4.105)\end{aligned}$$

which reduces to the previously obtained expression (4.98) for $N_{\rm f} = 0 = E_{\rm f}$ when only bsonic lines are present. In two space−time dimensions we find

$$\omega\,(G) = 2 - \frac{1}{2}\,E_{\rm f} - V\left(2 - \frac{1}{2}\,N_{\rm f}\right) \qquad [\,\text{two dimensions}\,]$$

which shows that $N_{\rm f} \leq 4$ for $D = 2$ otherwise the divergence would be growing with the number of vertices. It turns out that there is a restriction on the type of allowed fermion interactions at the quantum level even in one space and one time dimensions : it must be of a degree not higher than $\bar{\psi}_{\alpha}\,(x)\,\psi_{\beta}\,(y)\,\bar{\psi}_{\delta}\,(z)\,\psi_{\eta}\,(w)$. We can understand this fact from another point of view : unlike boson fields, which are dimensionless in two space−time dimensions, the spinor fields in $D = 2$ have canonical dimensions $\mathrm{eV}^{\,1/2}$, so that $(\bar{\psi}\psi)^2$ is the local monomial of the highest power that does not necessitate the introduction of a coupling parameter with an inverse mass power engeneering dimensions.

In four space−time dimensions we have instead

$$\omega\,(G) = 4 - \frac{3}{2}\,E_{\rm f} - E_{\rm b} - V\left(4 - \frac{3}{2}\,N_{\rm f} - N_{\rm b}\right) \qquad [\,\text{four dimensions}\,]$$

Now the necessary condition that prevents the number of the species of the primitively divergent graphs to grow up with the number of vertices yields

$$4 - \frac{3}{2}\,N_{\rm f} - N_{\rm b} \geq 0 \qquad (\,D = 4\,)$$

where $N_{\rm f}$ is even. The possible solutions are

- $N_{\rm f} = 2 \vee N_{\rm b} = 0$ which corresponds to a spinor mass term and not to an interaction vertex

- $N_{\rm f} = 0 \vee N_{\rm b} = 2\,,\,3\,,\,4$ which corresponds to a scalar or vectror mass term as well as to the cubic and quartic bosonic field interactions, the cubic one being ruled out even classically by the stability condition, *i.e.* by the request that the energy operator must be bounded from below

- $N_{\rm f} = 2 \vee N_{\rm b} = 1$, the only non−trivial new solution that gives

$$\omega\,(G) = 4 - \frac{3}{2}\,E_{\rm f} - E_{\rm b}$$

This new solution describes the only boson−fermion interaction allowed by the requirement of power counting renormalizability and turns out to be incredibly restrictive : it must involve one fermion−antifermion pair and one scalar or vector real field, which means that in four space−time dimensions spinor fields must appear only quadratically in the classical Lagrange density. Once again, this can be gathered because of the fact that in $D = 4$ the spinor field amplitudes have canonical dimensions $[\,\psi\,] = {\rm eV}^{3/2}$, while bosonic field amplitudes have $[\,\phi\,] = [\,A_\mu\,] = {\rm eV}$. Thus, the only non−trivial interaction of dimension four is the one with two spinor and one boson fields. If the discrete parity symmetry is there we come to the only two possible parity−even interactions

$$g\,\phi\,(x)\,\bar{\psi}\,(x)\,\psi\,(x) \qquad\qquad e\,A_\mu\,(x)\,\bar{\psi}\,(x)\,\gamma^{\,\mu}\,\psi\,(x)$$

that is, Yukawa meson field theory and quantum electrodynamics. This remarkable selection enormously simplifies the analysis about all the admissible quantum field theory models involving spinor fields.

In conclusion we can classify all the interacting quantum field theory models according to three categories which are characterized by the canonical engeneering dimension of the coupling parameter :

1. super−renormalizable ↔ coupling has positive mass dimensions
2. renormalizable ↔ coupling is dimensionless
3. non−renormalizable ↔ coupling has negative mass dimensions

4.2.2 Renormalization

Let me remind the one loop structure of the primitively divergent proper vertices of the self−interacting real scalar field theory in four space−time dimensions : using *e.g.* dimensional regularization with $D = 2\omega$, $\epsilon = 2 - \omega$ we find in momentum space

$$\begin{aligned} \widetilde{\Gamma}^{(2)}\,(k) &= k^2 - m^2 \left\{ 1 - \frac{\lambda}{32\,\pi^2}\,\left[\frac{1}{\epsilon} + \psi\,(2) - \ln\frac{m^2}{4\pi\mu^2} \right] \right\} \\ &+\ O\,(\,\lambda^2\,) \end{aligned}$$

$$\widetilde{\Gamma}^{(4)}(k_1,k_2,k_3,k_4) \;=\;$$
$$(-\,\mathrm{i}\lambda)\left\{1-\frac{3\lambda}{32\pi^2}\left[\frac{1}{\epsilon}+\psi(1)+2-\ln\frac{m^2}{4\pi\mu^2}-\frac{1}{3}\,A(\,s,t,u\,)\right]\right\}$$
$$+\,O(\,\lambda^3\,)$$

$$A(\,s,t,u\,)=\sum_{z\,=\,s\,,\,t\,,\,u}\left(\frac{4m^2}{z}-1\right)^{1/2}\operatorname{arcctg}\sqrt{\frac{4m^2}{z}-1}$$

$$s=(k_1+k_2)^2 \qquad t=(k_1+k_3)^2 \qquad u=(k_1+k_4)^2$$

Notice that the finite part of the above expressions is arbitrary, depending upon the free mass scale μ . The idea for the removal of the poles in $\epsilon=2-\omega$ order by order in perturbation theory is very simple : alter the Feynman rules at each order in such a manner to obtain a finite result for $\epsilon\ \to\ 0$. To start with consider the divergence of the kinetic term. Its infinity can be cast away by inserting a new Feynman rule indicated by

$$\text{—×—}\ \stackrel{\text{def}}{=}\ -\,\frac{\lambda m^2}{32\,\pi^2}\ \left[\frac{1}{\epsilon}+\mathrm{F}_1\left(\epsilon\,,\ \frac{m^2}{4\pi\mu^2}\right)\right] \tag{4.106}$$

where F_1 is an arbitrary dimensionless function analytic when $\epsilon\ \to\ 0$, the presence of which just endorses the arbitrariness inside this procedure. As a matter of fact, after a subtaction of an infinity the remaining finite part can be anything. Thus, if we now calculate the new kinetic term we find

$$\widetilde{\Gamma}^{(2)}_R(k)\ =\ k^2-m^2\left\{1-\frac{\lambda}{32\,\pi^2}\ \left[\psi(2)-\ln\frac{m^2}{4\pi\mu^2}-\mathrm{F}_1\left(\epsilon\,,\ \frac{m^2}{4\pi\mu^2}\right)\right]\right\}$$
$$+\ \ O(\,\lambda^2\,)$$

which is finite up to the order $O\,(\,\lambda\,)$ although arbitrary. The extra−term (4.106) is named a *mass counterterm*. Actually, it is crucial to realize that its dependence on fields in configuration space is the same as the classical mass term in the Lagrangian, *viz.*

$$\frac{1}{2}\,m^2\,\phi^2\,(x)\left\{1-\frac{\lambda}{32\,\pi^2}\ \left[\frac{1}{\epsilon}+\mathrm{F}_1\left(\epsilon\,,\ \frac{m^2}{4\pi\mu^2}\right)\right]\right\} \tag{4.107}$$

Let us now turn to the 4−point proper vertex. Again, to the aim of removing its simple pole $1/\epsilon$ let us introduce the new Feynman rule

$$\otimes\ \stackrel{\text{def}}{=}\ (\,-\,\mathrm{i}\lambda\,)\cdot\frac{3\lambda}{32\pi^2}\left[\frac{1}{\epsilon}+\mathrm{G}_1\left(\epsilon\,,\ \frac{m^2}{4\pi\mu^2}\right)\right] \tag{4.108}$$

where G_1 is another arbitrary dimensionless function analytic when $\epsilon \to 0$. This extra $O(\lambda^2)$ *coupling constant counterterm* precisely corresponds to a new self−interaction term

$$-\frac{\lambda}{4!}\int \mathrm{d}x\ \phi^4(x)\left\{1+\frac{3\lambda}{32\pi^2}\left[\frac{1}{\epsilon}+\mathrm{G}_1\left(\epsilon\,,\ \frac{m^2}{4\pi\mu^2}\right)\right]\right\} \tag{4.109}$$

with the very same dependence upon the fields of the classical potential. Moreover, its addition drives to a new, finite 4−point proper vertex

$$\begin{aligned}&\widetilde{\Gamma}_R^{(4)}(k_1,k_2,k_3,k_4) \ = \ (-\mathrm{i}\lambda) \ \times \\ &\left\{1-\frac{3\lambda}{32\pi^2}\left[-\mathrm{G}_1\left(\epsilon\,,\ \frac{m^2}{4\pi\mu^2}\right)+\psi(1)+2-\ln\frac{m^2}{4\pi\mu^2}-\frac{1}{3}A(s,t,u)\right]\right\} \\ &+O(\lambda^3)\end{aligned}$$

Before proceeding further on let me turn to the euclidean formulation in 2ω−dimensions in such a manner to always deal with absolutely convergent dimensionally regularized integrals. The Feynman rules have been obtained in section 1.5 and the transition from the euclidean space to the Minkowski space−time for a generic Feynman integral is quite simple. Let me consider in fact a Feynman diagram G with L loops, V vertices and I internal lines. In order to pass from Minkowski's to euclidean spaces we have to multiply each propagator and each vertex by a factor $(+i)$ and a further factor $(+i)$ arises for each loop integration owing to Wick rotation $d^{2\omega}\ell = i\,d^{2\omega}\ell_E$. In fact we can write simbolically

$$\begin{aligned} G_M &= \left(\mathrm{d}^{2\omega}\ell\right)^L\left(\frac{i}{k^2-m^2+i\varepsilon}\right)^I(-i\lambda)^V \\ &= \left(i\,\mathrm{d}^{2\omega}\ell_E\right)^L\left(\frac{-i}{k_E^2+m^2}\right)^I(-i\lambda)^V \\ &= i^{L+I+V}(-1)^I\ G_E(m^2-i\varepsilon) \end{aligned} \tag{4.110}$$

which means the following relation between the minkowskian and euclidean proper vertices : namely,

$$\widetilde{\Gamma}_M^{(n)}(k_1\,,\ \ldots\,,\ k_n\,;\ m^2)=i^{L+I+V}(-1)^I\ \widetilde{\Gamma}_E^{(n)}(\bar{k}_1\,,\ \ldots\,,\ \bar{k}_n\,;\ m^2-i\varepsilon)$$

$$\bar{k}_{\mu\jmath}=(\mathbf{k}_\jmath\,,\ k_{4\jmath}) \qquad k_\jmath^\mu=(\mathbf{k}_\jmath\,,\ k_\jmath^0) \qquad k_{0\jmath}=i\,k_{4\jmath} \qquad (\jmath=1,2,\ \ldots\ n)$$

It follows that in the euclidean formulation we have

$$\begin{aligned}\widetilde{\Gamma}^{(2)}(\bar{k}) &= \bar{k}^2+m^2\left\{1-\frac{\lambda}{32\pi^2}\ \left[\frac{1}{\epsilon}+\psi(2)-\ln\frac{m^2}{4\pi\mu^2}\right]\right\} \\ &+ \ O(\lambda^2)\end{aligned}$$

$$\widetilde{\Gamma}^{(4)}(\bar{k}_1, \bar{k}_2, \bar{k}_3, \bar{k}_4) =$$
$$(-\lambda)\left\{1 - \frac{3\lambda}{32\pi^2}\left[\frac{1}{\epsilon} + \psi(1) + 2 - \ln\frac{m^2}{4\pi\mu^2} - \frac{1}{3}A(s,t,u)\right]\right\}$$
$$+ O(\lambda^3)$$

$$A(s,t,u) = \sum_{z=s,t,u}\left(\frac{4m^2}{z} + 1\right)^{1/2}$$
$$\times \left[\ln\left(1 + \sqrt{\frac{4m^2}{z} + 1}\right) - \ln\left(-1 + \sqrt{\frac{4m^2}{z} + 1}\right)\right]$$
$$s = (\bar{k}_1 + \bar{k}_2)^2 \qquad t = (\bar{k}_1 + \bar{k}_3)^2 \qquad u = (\bar{k}_1 + \bar{k}_4)^2$$

where I have omitted the lower case suffix E for the sake of brevity because the euclidean nature of the proper vertices is apparent in the momentum dependence $\bar{k}_{\mu\jmath}$. Then the 1−loop euclidean counterterms become

$$—\!\!\times\!\!— \stackrel{\text{def}}{=} -\frac{\lambda m^2}{32\pi^2}\left[\frac{1}{\epsilon} + \mathrm{F}_1\left(\epsilon, \frac{m^2}{4\pi\mu^2}\right)\right] \tag{4.111}$$

$$\otimes \stackrel{\text{def}}{=} (-\lambda)\cdot\frac{3\lambda}{32\pi^2}\left[\frac{1}{\epsilon} + \mathrm{G}_1\left(\epsilon, \frac{m^2}{4\pi\mu^2}\right)\right] \tag{4.112}$$

and the corresponding renormalized finite proper vertices

$$\widetilde{\Gamma}^{(2)}_R(\bar{k}) = \bar{k}^2 + m^2\left\{1 - \frac{\lambda}{32\pi^2}\left[\psi(2) - \ln\frac{m^2}{4\pi\mu^2} - \mathrm{F}_1\right]\right\}$$
$$+ O(\lambda^2)$$

$$\widetilde{\Gamma}^{(4)}_R(\bar{k}_1, \bar{k}_2, \bar{k}_3, \bar{k}_4) =$$
$$(-\lambda)\left\{1 - \frac{3\lambda}{32\pi^2}\left[-\mathrm{G}_1 - \mathbf{C} + 2 - \ln\frac{m^2}{4\pi\mu^2} - \frac{1}{3}A(s,t,u)\right]\right\}$$
$$+ O(\lambda^3)$$

The 2−loop 2−point proper vertex reads

$$\widetilde{\Gamma}^{(2)}(\bar{k}) = \bar{k}^2\left(1 - \frac{\hat{\lambda}^2}{24}\cdot\frac{1}{\epsilon}\right)$$
$$+ m^2\left\{1 - \frac{1}{2}\hat{\lambda}\left[\psi(2) - \ln\widehat{m}^2 - \mathrm{F}_1\right]\right.$$
$$\left. + \frac{1}{2}\hat{\lambda}^2\left[\frac{1}{\epsilon^2} + \frac{1}{2\epsilon}(\mathrm{F}_1 + 3\mathrm{G}_1 - 1) + \cdots\right]\right\}$$
$$+ O(\lambda^3)$$

in which I have set $\hat{\lambda} = \lambda/16\pi^2$ and $\widehat{m}^2 = m^2/4\pi\mu^2$ while dots stand for the complicated finite part for $\epsilon \to 0$ that I have not explicitly shown. Hence the 2−loop mass counterterm reads

$$—\!\times\!— \stackrel{\text{def}}{=}$$

$$- \quad \frac{1}{2} m^2 \left\{ \frac{\hat{\lambda}^2}{\epsilon^2} + \frac{1}{\epsilon} \left[\hat{\lambda} + \frac{1}{4} \hat{\lambda}^2 (\mathrm{F}_1 + 3\,\mathrm{G}_1 - 1) \right] + \hat{\lambda}^2 \,\mathrm{F}_2 + \hat{\lambda}\,\mathrm{F}_1 \right\}$$

where $\mathrm{F}_2(\epsilon, \widehat{m}^2)$ is again some new arbitrary function that is finite for $\epsilon \to 0$.

The $O(\lambda^2)$ modified euclidean Lagrange density is provided by

$$\mathcal{L}_R = \mathcal{L}_E + \mathcal{L}_{\text{c.t.}} \tag{4.113}$$

in which I have set

$$\mathcal{L}_E = \frac{1}{2}\,\partial_\mu \phi_E\, \partial_\mu \phi_E + \frac{1}{2}\, m^2\, \phi_E^2 + \frac{\lambda}{4!}\, \phi_E^4$$

$$\mathcal{L}_{\text{c.t.}} \;=\; A\, \frac{1}{2} \partial_\mu \phi_E\, \partial_\mu \phi_E \;+\; B\, \frac{1}{2} m^2 \phi_E^2 \;+\; C\, \frac{\lambda}{4!}\, \phi_E^4 \tag{4.114}$$

where $\mathrm{H}_2(\epsilon, \widehat{m}^2)$ is arbitrary and analytic for $\epsilon \to 0$, whereas I have set

$$A = -\,\hat{\lambda}^2 \left(\frac{1}{24\,\epsilon} + \mathrm{H}_2 \right) \tag{4.115}$$

$$B = \frac{1}{2} \left\{ \frac{\hat{\lambda}^2}{\epsilon^2} + \frac{1}{\epsilon} \left[\hat{\lambda} + \frac{1}{4} \hat{\lambda}^2 (\mathrm{F}_1 + 3\,\mathrm{G}_1 - 1) \right] + \hat{\lambda}^2 \,\mathrm{F}_2 + \hat{\lambda}\,\mathrm{F}_1 \right\} \tag{4.116}$$

$$C = \frac{3\hat{\lambda}}{2} \left(\frac{1}{\epsilon} + \mathrm{G}_1 \right) \tag{4.117}$$

It is apparent that the $O(\lambda^2)$ modified euclidean Lagrange density does exactly share the very same structure of the classical euclidean Lagrange density $\mathcal{L}_E$ but for the especially tuned divergent coefficients A, B and C in such a manner that the ensuing Schwinger's functions are finite though arbitrary when the regularization is removed, *i.e.* for $\epsilon \to 0$. Turning back to the Minkowski space−time, after a redefinition of field amplitude, mass and self−coupling, we can recast the renormalized Lagrangian in the form

$$\mathcal{L}_R = \frac{1}{2}\,\partial_\mu \phi_0\, \partial^\mu \phi_0 \,-\, \frac{1}{2}\, m_0^2\, \phi_0^2 - \frac{\lambda_0}{4!}\, \phi_0^4 \tag{4.118}$$

with

$$\phi_0(x) = \sqrt{1+A} = Z_\phi^{1/2}\,\phi(x) \tag{4.119}$$

$$m_0^2 = m^2\,\frac{1+B}{1+A} = m^2\,(1+B)\,Z_\phi^{-1} \tag{4.120}$$

$$\lambda_0 = \lambda\,\mu^{2\epsilon}\,\frac{1+C}{1+2A+A^2} = \lambda\,\mu^{2\epsilon}\,(1+C)\,Z_\phi^{-2} \tag{4.121}$$

The quantities ϕ_0, m_0, λ_0, which are divergent for $\epsilon \to 0$, are called the *bare field, mass and coupling parameter* respectively. It is very important to gather that the renormalized Lagrangian $\mathcal{L}_R$ looks exactly the same as the classical Lagrangian $\mathcal{L}$ but for parameters and fields. Moreover, $\mathcal{L}_R$ leads to a finite theory while $\mathcal{L}$ does not. This fact indicates that we can always put all the infinities of perturbation theory inside ϕ_0, m_0, λ_0. The infinities are then absorbed by renormalization. The bare quantities do diverge for $\epsilon \to 0$, while the renormalized quantities ϕ, m, λ all give finite although arbitrary values when the regulators are removed, *i.e.* , for $\epsilon \to 0$. *The latter have to be identified with the physical parameters and fields of the theory.* Sometimes it is convenient to introduce another very popular and widely employed notation, *viz.*

$$\mathcal{L}_R = \frac{1}{2}\,Z_3\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\,Z_0\,m^2\,\phi^2 - Z_1\,\frac{\lambda}{4!}\,\phi^4 \tag{4.122}$$

in which $Z_3 \equiv Z_\phi$ while

$$\phi_0(x) = Z_3^{1/2}\,\phi(x) \tag{4.123}$$

$$m_0^2 = Z_0\,Z_3^{-1}\,m^2 = m^2\,Z_m \tag{4.124}$$

$$\lambda_0 = Z_1\,Z_3^{-2}\,\lambda = \lambda\,Z_\lambda \tag{4.125}$$

Notice that in the so called *minimal subtraction scheme* ($MS-$scheme), in which all the arbitrary analytic functions $\mathrm{F}_1 = \mathrm{G}_1 = \mathrm{F}_2 = \mathrm{H}_2 = \cdots \equiv 0$ are all set equal to zero to all orders by the very definition, we find

$$Z_0 = 1 + \frac{\hat{\lambda}}{2\,\epsilon} + \frac{\hat{\lambda}^2}{4}\left(\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\right) + O(\lambda^3) \tag{4.126}$$

$$Z_3 = Z_\phi = 1 - \frac{\hat{\lambda}^2}{24\epsilon} + O(\lambda^3) \tag{4.127}$$

$$Z_m = 1 + \frac{1}{2\epsilon}\left(\hat{\lambda} - \frac{5\hat{\lambda}^2}{12}\right) + \frac{\hat{\lambda}^2}{2\epsilon^2} + O(\lambda^3) \tag{4.128}$$

In the functional integral approach, one integrates over the field variable. Thus, its rescaling by Z_ϕ can be always reabsorbed provided one redefine the classical source accordingly, by introducing the bare source

$$J_0(x) = Z_\phi^{-1/2} J(x) \tag{4.129}$$

and the bare classical fields

$$\phi_{c\ell,0}(x) = Z_\phi\, \phi_{c\ell}(x) \tag{4.130}$$

Then, on the one hand, starting from the renormalized Lagrangian (4.118) and taking functional derivatives with respect to the bare source $J_0(x)$ or the bare classical fields $\phi_{c\ell,0}(x)$, we obtain the Green's functions or the proper vertices of perturbation theory, in which the parameters m and λ are replaced by the bare ones m_0 and λ_0. On the other hand, had we started instead from the renormalized Lagrangian (4.122) and taken functional derivatives with respect to the classical source $J(x)$, then we end up with the finite Green's functions and proper vertices. For the 1PI Green's functions this equality reads

$$\widetilde{\Gamma}_0^{(n)}(k_1,\ldots,k_n;\lambda_0,m_0,\epsilon) = Z_\phi^{-n/2}\,\widetilde{\Gamma}_R^{(n)}(k_1,\ldots,k_n;\lambda,m,\epsilon) \tag{4.131}$$

where $\widetilde{\Gamma}_R^{(n)}$ are finite as $\epsilon \to 0$. In this equation one can either understand the bare parameters m_0 and λ_0 as functions of the renormalized or *dressed parameters* m and λ, or even regard the bare parameters as independent ones. In the latter case, the dressed parameters are then functions of the bare ones. Hence, in so doing, it turns out that the left hand side of (4.131) does not depend on the mass scale μ, whilst the right hand side depends upon μ explicitly as well as implicitly through m and λ. Therefore, by differentiating the relationship (4.131) with respect to μ we eventually obtain a differential equation, wich is called the *renormalization group equation*, that summarizes the deep content of the renormalization procedure : namely,

$$\left(\mu\frac{\partial}{\partial\mu} + \mu\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}\frac{\partial}{\partial\lambda} + \mu\frac{\mathrm{d}m}{\mathrm{d}\mu}\frac{\partial}{\partial m} - \frac{n}{2}\mu\frac{\partial}{\partial\mu}\ln Z_\phi\right)\widetilde{\Gamma}_R^{(n)} = 0 \tag{4.132}$$

The beauty of this renormalization group equation is that it merely involves the renormalized 1PI Green's functions which are finite as long as $\epsilon \to 0$. The various derivatives come from the implicit dependence of the renormalized proper vertices upon μ through m and λ. Define the coefficients

$$\beta(\lambda, m/\mu, \epsilon) \equiv \mu \frac{\mathrm{d}\lambda}{\mathrm{d}\mu} \tag{4.133}$$

$$\gamma_m(\lambda, m/\mu, \epsilon) \equiv \tfrac{1}{2} \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \ln m^2 \tag{4.134}$$

$$\gamma_d(\lambda, m/\mu, \epsilon) \equiv \tfrac{1}{2} \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \ln Z_\phi \tag{4.135}$$

They are dimensionless and analytic for $\epsilon \to 0$.

Chapter 5

Appendices

5.1 Useful physical constants

Precisely known physical constants : W.-M. Yao *et al.,* Journal of Physics, G **33**, 1 (2006) and 2007 partial update for 2008, `http://pdg.lbl.gov`

Speed of light in vacuum	$c = 299\,792\,458$ m s^{-1}
Planck constant, reduced	$\hbar = h/2\pi = 1.054\,571\,68(18) \times 10^{-34}$ J s
	$= 6.582\,119\,15(56) \times 10^{-22}$ MeV s
electron charge magnitude	$e = 4.803\,204\,41(41) \times 10^{-10}$ esu
fine-structure constant	$\alpha = e^2/4\pi\hbar c = 7.297\,352\,568(24) \times 10^{-3}$
Fermi coupling constant	$G_F/(\hbar c)^3 = 1.166\,37(1) \times 10^{-5}$ GeV^{-2}
electron mass	$m_e = 0.510\,998\,918(44)$ MeV/c^2
proton mass	$m_p = 938.272\,029(80)$ MeV/c^2
Bohr radius ($\hbar/\alpha m_e c$)	$a_\infty = 0.529\,177\,2108(18) \times 10^{-10}$ m
e^- Compton wavelength	$\lambda_e = \hbar/m_e c = 3.861\,592\,678(26) \times 10^{-13}$ m
classical electron radius	$r_e = \alpha\lambda_e = 2.817\,940\,325(28) \times 10^{-13}$ cm
Thomson cross section	$\sigma_T = 8\pi r_e^2/3 = 0.665\,245\,873(13)$ barn

Conversion factors :

$\hbar c = 197.326\ 968(17)$ MeV fm
$(\hbar c)$ 1 cm^{-1} $\simeq$ 2×10^{-14} GeV
$(\hbar c)$ 1 GeV^{-1} = 0.1973 fm   1 fm $\simeq$ $(\hbar c)$ 5 GeV^{-1}
$(\hbar c)^2$ 1 GeV^{-2} = 0.3894 mbarn
1 barn = 10^{-28} m^2
$(\hbar c)$ 1 eV m^{-1} = 1.973×10^{-25} GeV^2
$(e\hbar c)$ 1 Tesla = $(e\hbar c)$ 10^4 Gauss = 5.916×10^{-25} GeV^2
1 unit of R $\equiv$ $4\pi\alpha^2/3E_{CM}^2 = (\hbar c)^2$ 86.8 nbarns (E_{CM} in GeV)$^{-2}$

5.2 Dimensional regularization

Here we list some useful identities concerning dimensional regularization. The Levi–Civita symbol in the four dimensional Minkowski space–time is normalized according to

$$\epsilon^{0123} = -\,\epsilon_{0123} \equiv 1 \tag{5.1}$$

in such a way that the following identity holds true in the four dimensional Minkowski space–time: namely,

$$\begin{aligned}\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu}^{\ \lambda\rho\sigma} &= g^{\nu\rho}g^{\alpha\lambda}g^{\beta\sigma} + g^{\alpha\rho}g^{\beta\lambda}g^{\nu\sigma} + g^{\beta\rho}g^{\nu\lambda}g^{\alpha\sigma} \\ &- g^{\nu\lambda}g^{\alpha\rho}g^{\beta\sigma} - g^{\alpha\lambda}g^{\beta\rho}g^{\nu\sigma} - g^{\beta\lambda}g^{\nu\rho}g^{\alpha\sigma}\end{aligned} \tag{5.2}$$

Concerning dimensional regularization, we collect here below the definitions and key properties [23] for the $2^\omega \times 2^\omega$ $\gamma-$matrices in a $2\omega-$dimensional space-time with a Minkowski signature

$$\gamma^{\mu} = \begin{cases} \bar{\gamma}^{\mu} & \mu = 0,1,2,3 \\ \hat{\gamma}^{\mu} & \mu = 4,\ldots,2\omega-4 \end{cases} \tag{5.3}$$

$$\{\bar{\gamma}^{\mu},\bar{\gamma}^{\nu}\} = 2\bar{g}^{\mu\nu}\,\mathbb{I} \qquad \{\hat{\gamma}^{\mu},\hat{\gamma}^{\nu}\} = 2\hat{g}^{\mu\nu}\,\mathbb{I} \qquad \{\bar{\gamma}^{\mu},\hat{\gamma}^{\nu}\} = 0 \tag{5.4}$$

$$\|\,\bar{g}\,\| = \mathrm{diag}\,(+,-,-,-) \qquad \|\,\hat{g}\,\| = -\,\hat{\mathbb{I}} \tag{5.5}$$

$$\gamma_5 \equiv i\bar{\gamma}^0\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3 \qquad \gamma_5^2 = \mathbb{I} \qquad \{\bar{\gamma}^{\mu},\gamma_5\} = 0 = [\,\hat{\gamma}^{\mu},\gamma_5\,] \tag{5.6}$$

where $\mathbb{I}$ denotes the identity $2^\omega \times 2^\omega$ square matrix, whereas $\hat{\mathbb{I}}$ denotes the identity matrix in the $2\omega - 4$ dimensional euclidean space. Taking all the above listed equations into account, it is not difficult to check the following trace formulæ :

$$\mathrm{tr}\,(\gamma^{\mu}\gamma^{\nu}) = g^{\mu\nu}\,\mathrm{tr}\,\mathbb{I} = 2^{\omega}\,g^{\mu\nu} \tag{5.7}$$

$$2^{-\omega}\mathrm{tr}\,\left(\gamma^{\kappa}\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu}\right) = g^{\kappa\lambda}\,g^{\mu\nu} - g^{\kappa\mu}\,g^{\lambda\nu} + g^{\kappa\nu}\,g^{\lambda\mu} \tag{5.8}$$

$$\begin{aligned}2^{-\omega}\mathrm{tr}\,\left(\gamma^{\kappa}\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right) &= g^{\kappa\lambda}\,g^{\mu\sigma}\,g^{\nu\rho} - g^{\kappa\lambda}\,g^{\mu\rho}\,g^{\nu\sigma} - g^{\kappa\mu}\,g^{\lambda\sigma}\,g^{\nu\rho} \\ &+ g^{\kappa\mu}\,g^{\lambda\rho}\,g^{\nu\sigma} + g^{\kappa\nu}\,g^{\lambda\sigma}\,g^{\mu\rho} - g^{\kappa\nu}\,g^{\lambda\rho}\,g^{\mu\sigma} \\ &+ g^{\lambda\mu}\,g^{\kappa\sigma}\,g^{\nu\rho} - g^{\lambda\mu}\,g^{\kappa\rho}\,g^{\nu\sigma} - g^{\lambda\nu}\,g^{\kappa\sigma}\,g^{\mu\rho} \\ &+ g^{\lambda\nu}\,g^{\kappa\rho}\,g^{\mu\sigma} - g^{\mu\nu}\,g^{\kappa\rho}\,g^{\lambda\sigma} + g^{\mu\nu}\,g^{\kappa\sigma}\,g^{\lambda\rho} \\ &+ g^{\kappa\nu}\,g^{\lambda\mu}\,g^{\rho\sigma} - g^{\kappa\mu}\,g^{\lambda\nu}\,g^{\rho\sigma} + g^{\kappa\lambda}\,g^{\mu\nu}\,g^{\rho\sigma} \\ \mathrm{tr}\,\left(\bar{\gamma}^{\kappa}\bar{\gamma}^{\lambda}\hat{\gamma}^{\mu}\hat{\gamma}^{\nu}\right) &= 2^{\omega}\,\bar{g}^{\kappa\lambda}\,\hat{g}^{\mu\nu} \\ \mathrm{tr}\,\left(\gamma_5\bar{\gamma}^{\mu}\bar{\gamma}^{\lambda}\bar{\gamma}^{\rho}\bar{\gamma}^{\nu}\right) &= -\,i\,2^{\omega}\epsilon^{\mu\lambda\rho\nu} \\ \mathrm{tr}\,\left(\gamma_5\bar{\gamma}^{\mu}\bar{\gamma}^{\lambda}\bar{\gamma}^{\rho}\bar{\gamma}^{\nu}\bar{\gamma}^{\sigma}\bar{\gamma}^{\tau}\right) &= i\,2^{\omega}\left(\epsilon^{\nu\sigma\tau\mu}\,\bar{g}^{\lambda\rho} + \epsilon^{\nu\sigma\tau\rho}\,\bar{g}^{\lambda\mu} + \epsilon^{\mu\lambda\rho\sigma}\,\bar{g}^{\nu\tau}\right) \\ &- i\,2^{\omega}\left(\epsilon^{\nu\sigma\tau\lambda}\,\bar{g}^{\mu\rho} + \epsilon^{\mu\lambda\rho\nu}\,\bar{g}^{\sigma\tau} + \epsilon^{\mu\lambda\rho\tau}\,\bar{g}^{\nu\sigma}\right)\end{aligned} \tag{5.9}$$

Traces involving an odd number of Dirac matrices do vanish.
Remark : in $d = 2n$, $n \in \mathbb{N}$, the standard representation of the Dirac matrices has dimension 2^n, whereas in the dimensional regularization the Dirac matrices are infinite–dimensional. Nevertheless, if we set $\mathrm{tr}\mathbf{1} \equiv f(\omega)$, it is not necessary to choose $f(\omega) = 2^\omega$. It is usually convenient to set $f(\omega) = f(2) = 4$, $\forall\omega \in \mathbf{C}$ [see J. Collins, *Renormalization*, Cambridge University Press (1984) p. 84]. We can definitely agree on that.

5.3 Glossary of dimensional regularization

General Feynman parametrisation formula

$$D_1^{-a_1} D_2^{-a_2} \cdots D_k^{-a_k} = \frac{\Gamma(a_1 + a_2 + \cdots + a_k)}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_k)}$$
$$\int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \cdots \int_0^1 \mathrm{d}x_k\, \delta(1 - x_1 - x_2 - \cdots - x_k)$$
$$(x_1 D_1 + x_2 D_2 + \cdots + x_k D_k)^{-a_1 - a_2 - \cdots - a_k} \tag{5.10}$$

Definitions :

$$R(x,a) = x^2 - x + \frac{m^2}{k^2} = x^2 - x + a \qquad \Delta = -1 + \frac{4m^2}{k^2}$$

$$\int_p = \mu^{4-2\omega} \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}}$$

$$I(r, s) = \int_p (p^2 - m^2 + i\varepsilon)^{-r} [(p-k)^2 - m^2 + i\varepsilon]^{-s}$$

$$I^{\mu}(r, s) = \int_p \frac{p^\mu}{(p^2 - m^2 + i\varepsilon)^r [(p-k)^2 - m^2 + i\varepsilon]^s}$$

$$I^{\mu\nu}(r, s) = \int_p \frac{p^\mu p^\nu}{(p^2 - m^2 + i\varepsilon)^r [(p-k)^2 - m^2 + i\varepsilon]^s}$$

$$I^{\mu\nu\rho}(r, s) = \int_p \frac{p^\mu p^\nu p^\rho}{(p^2 - m^2 + i\varepsilon)^r [(p-k)^2 - m^2 + i\varepsilon]^s}$$

$$I^{\mu\nu\rho\sigma}(r, s) = \int_p \frac{p^\mu p^\nu p^\rho p^\sigma}{(p^2 - m^2 + i\varepsilon)^r [(p-k)^2 - m^2 + i\varepsilon]^s}$$

Parametric integrals

$$\int_0^1 \frac{dx}{R} = \frac{4}{\sqrt{\Delta}} \operatorname{arcctg}\sqrt{\Delta} \qquad \text{for} \quad 0 < k^2 < 4m^2 \tag{5.11}$$
$$= -4 \qquad \text{for} \quad k^2 = 4m^2$$
$$= \frac{-4}{\sqrt{-\Delta}} \operatorname{Arcth}\sqrt{-\Delta}$$
$$= \frac{2}{\sqrt{-\Delta}} \ln \frac{\sqrt{-\Delta} - 1}{1 + \sqrt{-\Delta}} \qquad \text{for} \quad k^2 > 4m^2 \vee k^2 < 0$$

$$\begin{aligned}
I_0 &\equiv \int_0^1 dx\ \ln\left(\frac{4\pi\mu^2}{R\,k^2}\right) \\
&= \ln\frac{4\pi\mu^2}{k^2} - \int_0^1 dx\ \ln R \\
&= \ln\frac{4\pi\mu^2}{m^2} + \int_0^1 dx\ \frac{2x^2 - x}{R} \\
&= 2 + \ln\frac{4\pi\mu^2}{m^2} + \frac{1}{2}(1-4a)\int_0^1 \frac{dx}{R} \qquad (5.12) \\
I_1 &\equiv \int_0^1 dx\ x\ln\left(\frac{4\pi\mu^2}{R\,k^2}\right) \\
&= 1 + \frac{1}{2}\ln\frac{4\pi\mu^2}{m^2} + \frac{1}{4}(1-4a)\int_0^1 \frac{dx}{R} \qquad (5.13) \\
I_2 &\equiv \int_0^1 dx\ x(1-x)\ln\left(\frac{4\pi\mu^2}{R\,k^2}\right) \\
&= \frac{1}{6}\ln\frac{4\pi\mu^2}{m^2} - \frac{1}{6}\int_0^1 \frac{dx}{R}\ (4x^4 - 8x^3 + 3x^2) \\
&= \frac{5}{18} + \frac{1}{6}\ln\frac{4\pi\mu^2}{m^2} + \frac{2a}{3} + \frac{1-2a-8a^2}{12}\int_0^1 \frac{dx}{R} \qquad (5.14) \\
& a = m^2/k^2
\end{aligned}$$

Scalar integrals

$$I(2,0) = I(0,2) = \frac{i}{16\pi^2}\,\Gamma(2-\omega)\left(\frac{4\pi\mu^2}{m^2}\right)^{2-\omega} \qquad (5.15)$$

$$\lim_{\omega\to 2} m^2\, I(3,0) = -\frac{i}{32\pi^2} \qquad (5.16)$$

$$\begin{aligned}
I(1,1) &= \frac{i}{16\pi^2}\,\Gamma(2-\omega)\int_0^1 dx\ \left(\frac{4\pi\mu^2}{R\,k^2}\right)^{2-\omega} \\
&\doteq \frac{i}{16\pi^2}\left\{\frac{1}{\epsilon} - \mathbf{C} + I_0\right\} \qquad (5.17)
\end{aligned}$$

$$I(2,1) = I(1,2) = -\frac{i}{16\pi^2}\int_0^1 \frac{dx}{R\,k^2}\ x \qquad (5.18)$$

$$I(2,2) = \frac{i}{16\pi^2}\int_0^1 \frac{dx}{[\,R\,k^2\,]^2}\ x(1-x) \qquad (5.19)$$

$$I(3,1) = I(1,3) = \frac{i}{32\pi^2}\int_0^1 \frac{dx}{[\,R\,k^2\,]^2}\ x^2 \qquad (5.20)$$

Vector integrals

$$
\begin{aligned}
I_\nu(1,1) &= \frac{i}{16\pi^2}\,k_\nu\,\Gamma(2-\omega)\int_0^1 dx\; x\left(\frac{4\pi\mu^2}{R\,k^2}\right)^{2-\omega} \\
&\doteq \frac{i\,k_\nu}{32\pi^2}\left\{\frac{1}{\epsilon}-\mathbf{C}+2I_1\right\} &(5.21)\\
I_\nu(2,1) &= -\frac{ik_\nu}{16\pi^2}\int_0^1 \frac{dx}{R\,k^2}\; x(1-x) &(5.22)\\
I_\nu(1,2) &= -\frac{ik_\nu}{16\pi^2}\int_0^1 \frac{dx}{R\,k^2}\; x^2 &(5.23)\\
I_\nu(2,2) &= \frac{ik_\nu}{16\pi^2}\int_0^1 \frac{dx}{[\,R\,k^2\,]^2}\; x^2\,(1-x) &(5.24)\\
I_\nu(3,1) &= \frac{ik_\nu}{32\pi^2}\int_0^1 \frac{dx}{[\,R\,k^2\,]^2}\; x^2(1-x) &(5.25)\\
I_\nu(1,3) &= \frac{ik_\nu}{32\pi^2}\int_0^1 \frac{dx}{[\,R\,k^2\,]^2}\; x^3 &(5.26)
\end{aligned}
$$

Rank two tensor integrals

$$
\begin{aligned}
I_{\lambda\rho}(2,0) &= -\frac{i}{16\pi^2}\;m^2\,g_{\lambda\rho}\,\frac{\Gamma(2-\omega)}{2-2\omega}\left(\frac{4\pi\mu^2}{m^2}\right)^{2-\omega} &(5.27)\\
I_{\lambda\rho}(0,2) &= I_{\lambda\rho}(2,0)\;+\;k_\lambda k_\rho\,I(2,0) &(5.28)\\
I_{\lambda\rho}(3,0) &= \frac{i}{16\pi^2}\;g_{\lambda\rho}\,\frac{\Gamma(2-\omega)}{2\cdot 2!}\left(\frac{4\pi\mu^2}{m^2}\right)^{2-\omega} &(5.29)\\
\lim_{\omega\to 2}\;m^2\,I_{\lambda\rho}(4,0) &= -\frac{i}{192\pi^2}\;g_{\lambda\rho} &(5.30)\\
I_{\lambda\nu}(1,1) &= \frac{i}{16\pi^2}\,\Gamma(2-\omega)\int_0^1 dx\;\left(\frac{4\pi\mu^2}{R\,k^2}\right)^{2-\omega}\\
&\times\;\left\{x^2k_\lambda\,k_\nu-\frac{g_{\lambda\nu}}{2\omega-2}\left[\,x(1-x)k^2-m^2\,\right]\right\}\\
&\doteq \frac{i}{48\pi^2}\,k_\lambda\,k_\nu\left\{\frac{1}{\epsilon}-\mathbf{C}+3\,[\,I_1-I_2\,]\right\}\\
&-\;\frac{i}{32\pi^2}\,k^2g_{\lambda\nu}\left\{\frac{1}{6\epsilon}-\frac{\mathbf{C}}{6}+\frac{1}{6}+I_2\right\}\\
&+\;\frac{i}{32\pi^2}\,m^2g_{\lambda\nu}\left\{\frac{1}{\epsilon}-\mathbf{C}+1+I_0\right\} &(5.31)
\end{aligned}
$$

$$
\begin{aligned}
I^{\lambda\nu}(2,1) &= \frac{i}{32\pi^2}\, g^{\lambda\nu}\, \Gamma(2-\omega) \int_0^1 dx\; x \left(\frac{4\pi\mu^2}{R\,k^2}\right)^{2-\omega} \\
&- \frac{i}{16\pi^2}\, k^\lambda\, k^\nu \int_0^1 \frac{dx}{R\,k^2}\, x^2\,(1-x) \\
&\doteq \frac{i}{64\pi^2}\, g^{\lambda\nu} \left\{\frac{1}{\epsilon} - \mathbf{C} + 2I_1\right\} - \frac{i}{16\pi^2}\, k^\lambda\, k^\nu \int_0^1 \frac{dx}{Rk^2}\,(1-x)\,x^2
\end{aligned} \tag{5.32}
$$

$$
\begin{aligned}
I^{\lambda\nu}(1,2) &= \frac{i}{32\pi^2}\, g^{\lambda\nu}\, \Gamma(2-\omega) \int_0^1 dx\; x \left(\frac{4\pi\mu^2}{R\,k^2}\right)^{2-\omega} \\
&- \frac{i}{16\pi^2}\, k^\lambda\, k^\nu \int_0^1 \frac{dx}{R\,k^2}\, x^3 \\
&\doteq \frac{i}{64\pi^2}\, g^{\lambda\nu} \left\{\frac{1}{\epsilon} - \mathbf{C} + 2I_1\right\} - \frac{i}{16\pi^2}\, k^\lambda\, k^\nu \int_0^1 \frac{dx}{Rk^2}\, x^3
\end{aligned} \tag{5.33}
$$

$$
I^{\lambda\nu}(2,2) = \frac{-i}{32\pi^2}\left\{ g^{\lambda\nu} \int_0^1 \frac{dx}{Rk^2}\, x(1-x) - 2k^\lambda k^\nu \int_0^1 \frac{dx}{[\,Rk^2\,]^2}\, x^3\,(1-x)\right\} \tag{5.34}
$$

$$
I^{\lambda\nu}(3,1) = \frac{-i}{64\pi^2}\left\{ g^{\lambda\nu} \int_0^1 \frac{dx}{Rk^2}\, x^2 - 2k^\lambda k^\nu \int_0^1 \frac{dx}{[\,Rk^2\,]^2}\, x^2(1-x)^2\right\} \tag{5.35}
$$

Rank three tensor integrals

$$
\begin{aligned}
I^{\lambda\nu\rho}(1,1) &= \frac{i}{16\pi^2}\, \Gamma(2-\omega) \int_0^1 dx \left(\frac{4\pi\mu^2}{R\,k^2}\right)^{2-\omega} (2\omega-2)^{-1} \\
&\times \left\{ \left[\, xm^2 - x^2(1-x)k^2\,\right] (g^{\lambda\nu}\, k^\rho + g^{\nu\rho}\, k^\lambda + g^{\rho\lambda}\, k^\nu) \right. \\
&+ \left. (1-x)^3 k^\lambda\, k^\nu\, k^\rho \right\}
\end{aligned} \tag{5.36}
$$

$$
\begin{aligned}
I^{\lambda\nu\rho}(2,2) &= \frac{i}{32\pi^2}\left\{ 2\,k^\lambda\, k^\nu\, k^\rho \int_0^1 \frac{dx}{[\,R\,k^2\,]^2}\; x^4\,(1-x) \right. \\
&- \left. (g^{\lambda\nu}\, k^\rho + g^{\nu\rho}\, k^\lambda + g^{\rho\lambda}\, k^\nu) \int_0^1 \frac{dx}{R\,k^2}\; x^2\,(1-x) \right\}
\end{aligned} \tag{5.37}
$$

$$
\begin{aligned}
I^{\lambda\nu\rho}(3,1) &= \frac{i}{64\pi^2}\left\{2\,k^\lambda\,k^\nu\,k^\rho \int_0^1 \frac{dx}{[\,R\,k^2\,]^2}\;x^3\,(1-x)^2\right. \\
&- \left.(g^{\lambda\nu}\,k^{\,\rho} + g^{\nu\rho}\,k^\lambda + g^{\rho\lambda}\,k^\nu)\int_0^1 \frac{dx}{R\,k^2}\;x^2\,(1-x)\right\} \qquad (5.38)
\end{aligned}
$$

Rank four tensor integrals

$$
\begin{aligned}
I_{\lambda\rho\sigma\tau}(4,0) &= \frac{i}{384\pi^2}\,\Gamma(2-\omega)\left(\frac{4\pi\mu^2}{m^2}\right)^{2-\omega} \\
&\times\;(g_{\lambda\rho}\,g_{\sigma\tau} + g_{\lambda\sigma}\,g_{\tau\rho} + g_{\lambda\tau}\,g_{\rho\sigma})\;\;; \qquad (5.39) \\
I^{\lambda\nu\rho\sigma}(2,2) &= \frac{i}{64\pi^2}\,\Gamma(2-\omega)\;\left(g^{\lambda\nu}\,g^{\rho\sigma} + g^{\nu\rho}\,g^{\lambda\sigma} + g^{\rho\lambda}\,g^{\nu\sigma}\right) \\
&\times\;\int_0^1 dx\;x(1-x)\left(\frac{4\pi\mu^2}{R\,k^2}\right)^{2-\omega} \\
&+\;\frac{i}{16\pi^2}\,k^\lambda\,k^\nu\,k^{\,\rho}\,k^\sigma\int_0^1 \frac{dx}{[\,R\,k^2\,]^2}\;x^5\,(1-x) \\
&-\;\left(g^{\nu\lambda}\,k^{\,\rho}\,k^\sigma + \text{ cycl. perm.}\right)\;\frac{i}{32\pi^2}\int_0^1 \frac{dx}{R\,k^2}\;x^3\,(1-x) \\
&\doteq\;\frac{i}{384\pi^2}\,(g^{\lambda\nu}\,g^{\rho\sigma} + g^{\nu\rho}\,g^{\lambda\sigma} + g^{\rho\lambda}\,g^{\nu\sigma})\left\{\frac{1}{\epsilon} - \gamma + 6I_2(\xi)\right\} \\
&+\;\frac{i}{16\pi^2}\,k^\lambda\,k^\nu\,k^{\,\rho}\,k^\sigma\int_0^1 \frac{dx}{[\,R\,k^2\,]^2}\;x^5\,(1-x) \\
&-\;\left(g^{\nu\lambda}\,k^{\,\rho}\,k^\sigma + \text{ cycl. perm.}\right) \\
&\times\;\frac{i}{32\pi^2}\int_0^1 \frac{dx}{R\,k^2}\;x^3\,(1-x) \qquad (5.40)
\end{aligned}
$$

$$
\begin{aligned}
I^{\lambda\nu\rho\sigma}(3,1) &= \frac{i}{128\pi^2}\,\Gamma(2-\omega)\;\left(g^{\lambda\nu}\,g^{\rho\sigma} + g^{\nu\rho}\,g^{\lambda\sigma} + g^{\rho\lambda}\,g^{\nu\sigma}\right) \\
&\times\;\int_0^1 dx\;x^2\left(\frac{4\pi\mu^2}{R\,k^2}\right)^{2-\omega} \\
&+\;\frac{i}{32\pi^2}\,k^\lambda\,k^\nu\,k^{\,\rho}\,k^\sigma\int_0^1 \frac{dx}{[\,R\,k^2\,]^2}\;x^4\,(1-x)^2 \\
&-\;\left(g^{\nu\lambda}\,k^{\,\rho}\,k^\sigma + \text{ cycl. perm.}\right)\;\frac{i}{64\pi^2}\int_0^1 \frac{dx}{R\,k^2}\;x^2(1-x)^2
\end{aligned}
$$

$$
\begin{aligned}
\doteq \quad & \frac{i}{384\pi^2}\,(g^{\lambda\nu}\,g^{\rho\sigma} + g^{\nu\rho}\,g^{\lambda\sigma} + g^{\rho\lambda}\,g^{\nu\sigma}) \\
\times \quad & \left\{ \frac{1}{\epsilon} - \mathbf{C} + 3I_1 - 3I_2 \right\} \\
+ \quad & \frac{i}{32\pi^2}\,k^\lambda\,k^\nu\,k^\rho\,k^\sigma \int_0^1 \frac{dx}{[\,R\,k^2\,]^2}\;x^4(1-x)^2 \\
- \quad & \big(g^{\nu\lambda}\,k^\rho\,k^\sigma + \text{ cycl. perm.}\big) \int_0^1 \frac{dx}{R\,k^2}\;x^2(1-x)^2 \qquad (5.41)
\end{aligned}
$$

Bibliography

[1] John David Jackson, *Classical Electrodynamics*, John Wiley & Sons, New York, 1962.

[2] L.D. Landau, E.M. Lifšits, *Teoria dei campi*, Editori Riuniti/Edizioni Mir, Roma, 1976.

[3] Roberto Soldati, *Note on line del corso di meccanica statistica*, http://www.robertosoldati.com, 2007.

[4] E. Merzbacher, *Quantum Mechanics. Second edition*, John Wiley & Sons, New York, 1970 ;
L.D. Landau and E.M. Lifšits, *Meccanica Quantistica. Teoria non relativistica*, Editori Riuniti/Edizioni Mir, Roma, 1976 ;
C. Cohen–Tannoudji, B. Diu and F. Laloë, *Mécanique quantique I*, Hermann, Paris, 1998.

[5] G.Ya. Lyubarskii, *The Application of Group Theory in Physics*, Pergamon Press, Oxford, 1960 ;
Mark Naïmark and A. Stern, *Théorie des représentations des groupes*, Editions Mir, Moscou, 1979.

[6] Eugene Paul Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektrum*, Fredrick Vieweg und Sohn, Braunschweig, Deutschland, 1931, pp. 251–254 ; *Group Theory and its Application to the Quantum Theory of Atomic Spectra*, Academic Press Inc., New York, 1959, pp. 233–236.

[7] Valentine Bargmann and E.P. Wigner, *Group theoretical discussion of relativistic wave equations*, Proceedings of the National Academy of Sciences, Vol. **34**, No. 5, 211–223 (May 1948); *Relativistic Invariance and Quantum Phenomena*, Review of Modern Physics **29**, 255–268 (1957).

[8] N.N. Bogoljubov and D.V. Širkov, *Introduction to the Theory of Quantized Fields*, Interscience Publishers, New York, 1959.

[9] V.B. Berestetskij, E.M. Lifšits and L.P. Pitaevskij, *Teoria quantistica relativistica*, Editori Riuniti/Edizioni Mir, Roma, 1978.

[10] R.J. Rivers, *Path integral methods in quantum field theory*, Cambridge University Press, Cambridge (UK), 1987.

[11] Sidney R. Coleman, *The Uses of Instantons*, Proceedings of the 1977 International School of Subnuclear Physics, Erice, Antonino Zichichi Editor, Academic Press, New York, 1979.

[12] Claude Itzykson and Jean-Bernard Zuber, *Quantum Field Theory*, McGraw-Hill, New York, 1980.

[13] Pierre Ramond, *Field Theory: A Modern Primer*, Benjamin, Reading, Massachusetts, 1981.

[14] Ian J.R. Aitchison and Anthony J.G. Hey, *Gauge theories in particle physics. A practical introduction*, Adam Hilger, Bristol, 1982.

[15] B. de Wit, J. Smith, *Field Theory in Particle Physics – Volume 1*, Elsevier Science Publisher, Amsterdam, 1986.

[16] Stefan Pokorski, *Gauge field theories*, Cambridge University Press, Cambridge (UK), 1987.

[17] Michel Le Bellac, *Des phénomènes critiques aux champs de jauge. Une introduction aux méthodes et aux applications de la théorie quantique des champs*, InterEditions/Editions du CNRS, 1988.

[18] Lowell S. Brown, *Quantum Field Theory*, Cambridge University Press, Cambridge, United Kingdom, 1992.

[19] M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Perseus Books, Reading, Massachusetts, 1995.

[20] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, New York, 1978.

[21] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products*, Fifth Edition, Alan Jeffrey Editor, Academic Press, San Diego, 1996.

[22] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher trascendental functions*, Volume I, Bateman Manuscript Project, McGraw-Hill, New York, 1953.

[23] G. 't Hooft and M.J.G. Veltman, *Regularization and renormalization of gauge fields*, Nucl. Phys. B**44**, 189 (1972);
P. Breitenlohner and D. Maison, *Dimensional renormalization and the action principle*, Comm. Math. Phys. **52**, 11 (1977);
Dimensionally renormalized Green's functions for theories with massless particles, 1., Comm. Math. Phys. **52**, 39 (1977);
J. Collins, *Renormalization*, Cambridge University Press, Cambridge, United Kingdom, 1984.

[24] Hendrik Brugt Gerhard Casimir, *Zur Intensität der Streustrahlung gebundener Elektronen*, Helvetica Physica Acta **6**, 287–304 (1933).

[25] I.E. Tamm, *On Free Electron Interaction with Radiation in the Dirac Theory of the Electron and in Quantum Electrodynamics*, Zeitschrift der Physik **62**, 545 (1930).

[26] Vladimir A. Fock, *Konfigurationsraum und zweite quantelung*, Zeitschrift der Physik A **75**, 622–647 (1932).