09520 - TEORIA DEI CAMPI (Field Theory)

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Assessment tests

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Contents

0.1 *Monday 1 July 2019* **1. QFT1 Medley 2. QFT2 Medley**

QUANTUM FIELD THEORY **1.**

1. Consider the classical radiation field and write the Lagrangian and the whole set of the field equations. Show that its total angular momentum density third rank tensor can always be cast into a purely orbital form and comment the result. Provide some good reasons why the photon spin is customarily set equal to 1.
2. Write the normal modes expansion of a charged scalar Klein-Gordon quantum field. Derive the corresponding energy-momentum and charge operators, both in terms of the operator valued tempered distribution local field and of the creation-destruction operators.
3. Consider a Weyl spinor field. Write the classical Action - in both the two and four component forms of the Weyl spinor field - and list all its symmetries. Derive and solve the field equations. Find the normal modes expansions for the corresponding quantum field, which describes non interacting mass-less neutrino and anti-neutrino particles, specifying the algebra of the creation-destruction operators.

QUANTUM FIELD THEORY **2.**

Obtain the QED fermion effective Action, i.e. the Dirac spinor functional determinant in the presence of a classical real vector field, in perturbation theory. Write its explicit expression in the case of a uniform, i.e. constant and homogeneous, background real vector field.

0.2 *Monday 3 June 2019* 1. QFT1 Medley 2. QFT2 Medley

QUANTUM FIELD THEORY **1.**

1. A rotation of an orthogonal left handed reference frame in the three dimensional Euclidean space can be represented as a product of three orthogonal matrices: the rotation matrix $R_3(\varphi)$ about the OZ axis, the rotation matrix $R_1(\theta)$ about the OX' axis, which is called the nodal line, and the rotation matrix $R_3(\psi)$ about the OZ' axis, *i.e.*
$$R(g) = R(\varphi, \theta, \psi) = R_3(\psi)\, R_1(\theta)\, R_3(\varphi)$$
where $0 \leq \varphi < 2\pi$; $0 \leq \theta \leq \pi$; $0 \leq \psi < 2\pi$ are the Euler angles. Write any element of the fundamental representation $\boldsymbol{\tau}_{\frac{1}{2}}(\varphi, \theta, \psi)$ as a function of the Euler angles and verify that $\boldsymbol{\tau}_{\frac{1}{2}}(0, \theta, \psi) = -\,\boldsymbol{\tau}_{\frac{1}{2}}(2\pi, \theta, \psi)$. What does it mean?

2. Find the Fourier transform of the adjoint spinor propagator $\bar{S}^{\,F}(y-x)$, *i.e.* the solution of the non-homogeneous adjoint Dirac equation
$$i\delta\,(x-y) = \bar{S}^{\,F}(\,y-x\,)\,(\,i\overleftarrow{\partial\!\!\!/}_x + M\,)$$
which fulfills causality.

3. Consider the vector potential of the radiation field in the Feynman gauge, which satisfies the canonical commutation relations
$$\left[\, A^{\lambda}(x)\,,\, A^{\nu}(y)\,\right] = ig^{\lambda\nu}\, D_0(x-y)$$
$$[\, B(x)\,,\, A^{\nu}(y)\,] = i\partial_x^{\nu}\, D_0(x-y) \qquad [\, B(x)\,,\, B(y)\,] = 0$$
where $B(x)$ is the auxiliary scalar field, whereas the mass-less Pauli-Jordan real and odd distribution reads
$$D_0(x) = D_0^{(-)}(x) + D_0^{(+)}(x) = \lim_{m\to 0} D(x\,;\,m)$$
$$D_0(x-y) \equiv \frac{1}{i}\int \frac{\mathrm{d}^4k}{(2\pi)^3}\;\delta\left(k^2\right)\,\mathrm{sgn}\,(k_0)\;\exp\{-\,ik\cdot(x-y)\}$$
Calculate the commutator $\left[\, F^{\,\rho\lambda}(x)\,,\, B(y)\,\right]$ and comment its relation with the auxiliary condition $B^{\,(-)}(x)|\mathrm{phys}\rangle = 0$

Calculate the commutators $[\, F^{\,\mu\nu}(x)\,,\, F^{\,\rho\sigma}(y)\,]$ and comment the results for space-like separations $(x-y)^2 < 0$

QUANTUM FIELD THEORY **2.**

1. Consider a non-Abelian gauge theory of gauge group SU(N). Show that the field strength transforms according to the adjoint representation. Explain why the adjoint representation of SU(N) is real.

2. Consider two different Nucleons within the context of the Yukawa Model. Write the collision matrix up to the second order in the Yukawa coupling $y > 0$. Obtain the proton-neutron scattering amplitude, up to the same approximation, from the LSZ reduction formulas.

3. Consider the $e^- \mu^-$ scattering in QED, up to the lowest order in the fine structure constant. Write the Feynman rules and derive the lowest order scattering amplitude for the above process. For energies of the colliding particles much smaller of electron rest energy, show that the interaction is described by the repulsive Coulomb potential.

Solution.

QUANTUM FIELD THEORY **1.**

1. $SU(2)$ provides a fundamental two dimensional representation of the abstract rotation group, the Hermitean generators of which being given by $\frac{1}{2}\sigma_a$ $(a = 1, 2, 3)$, so that the element corresponding to $R_3(\varphi)$ is evidently

$$\boldsymbol{\tau}_{\frac{1}{2}}(\varphi) = \exp\{(i/2)\,\sigma_3\,\varphi\} = \mathbb{I}\,\cos\tfrac{1}{2}\varphi + i\sigma_3\,\sin\tfrac{1}{2}\varphi = \begin{pmatrix} \mathrm{e}^{\,i\varphi/2} & 0 \\ 0 & e^{-\,i\varphi/2} \end{pmatrix}$$

Quite analogously, the $SU(2)$ element corresponding to a rotation around the nodal line can be written as

$$\boldsymbol{\tau}_{\frac{1}{2}}(\theta) = \exp\{(i/2)\,\sigma_1\,\theta\} = \mathbb{I}\,\cos\tfrac{1}{2}\theta + i\sigma_1\,\sin\tfrac{1}{2}\theta = \begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

Thus we eventually obtain

$$\begin{aligned} \boldsymbol{\tau}_{\frac{1}{2}}(\varphi,\theta,\psi) &= g_3(\psi)g_1(\theta)g_3(\varphi) \\ &= \begin{pmatrix} \mathrm{e}^{\,i(\varphi+\psi)/2}\cos\theta/2 & i\mathrm{e}^{\,i(\psi-\varphi)/2}\sin\theta/2 \\ i\mathrm{e}^{\,i(\varphi-\psi)/2}\sin\theta/2 & \mathrm{e}^{-\,i(\varphi+\psi)/2}\cos\theta/2 \end{pmatrix} \end{aligned}$$

in such a manner that, for instance,

$$\boldsymbol{\tau}_{\frac{1}{2}}(0,\theta,\psi) = -\,\boldsymbol{\tau}_{\frac{1}{2}}(2\pi,\theta,\psi)$$

Hence, the representation $\boldsymbol{\tau}_{\frac{1}{2}}(\varphi,\theta,\psi)$ is a unitary, irreducible, continuous albeit double-valued representation of the rotation group. This entails that the proper rotation group $SO(3)$ is not simply connected.

2. The Fourier representation of the Feynman propagator for the Dirac spinor quantum field reads

$$\begin{aligned} S^F(x-y) &= (i\partial\!\!\!/_x + M)\,D_F(x-y) \\ &= \frac{i}{(2\pi)^4}\int \mathrm{d}^4p\;\frac{p\!\!\!/ + M}{p^2 - M^2 + i\varepsilon}\;\exp\{-\,ip\cdot(x-y)\} \\ &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4p\;\left(\frac{i}{p\!\!\!/ - M}\right)\;\exp\{-\,ip\cdot(x-y)\} \end{aligned}$$

where

$$\frac{i(\,p\!\!\!/ + M\,)_{\alpha\beta}}{p^2 - M^2 + i\varepsilon} = \left(\frac{i}{p\!\!\!/ - M}\right)_{\alpha\beta}$$

The spinor propagator fulfills the non-homogeneous matrix-type differential equation

$$(i\partial\!\!\!/_x - M)_{\alpha\beta}\, S^F_{\beta\gamma}(x-y) \;=\; i\delta(x-y)\,\delta_{\gamma\alpha}$$

Then it is easy to write the adjoint form of the non-homogeneous equation for the Feynman propagator of the spinor field. To this concern, let us first obtain the Hermitean conjugate of previous equation *viz.*,

$$\begin{aligned} i\delta(x-y) \;&=\; i(\partial/\partial x^\mu)\, S^\dagger_F(x-y)\,\gamma^{\mu\,\dagger} + M\, S^\dagger_F(x-y) \\ &=\; i(\partial/\partial x^\mu)\, S^\dagger_F(x-y)\,\gamma^0\,\gamma^\mu\,\gamma^0 + M\, S^\dagger_F(x-y) \end{aligned}$$

Multiplication by γ^0 from left and right yields

$$\begin{aligned} i\delta(x-y) \;&=\; i\gamma^0\,(\partial/\partial x^\mu)\, S^\dagger_F(x-y)\,\gamma^0\,\gamma^\mu + \gamma^0\, M\, S^\dagger_F(x-y)\,\gamma^0 \\ &\stackrel{\mathrm{def}}{=}\; \bar{S}^F(\,y-x)\,(\,i\overleftarrow{\partial\!\!\!/}_x + M\,) \end{aligned}$$

where

$$\begin{aligned} \bar{S}^F(\,y-x) \;&=\; \gamma^0\, S^\dagger_F(x-y)\,\gamma^0 \\ =\;& \frac{-\,i}{(2\pi)^4}\int \mathrm{d}^4p\; \frac{p\!\!\!/ + M}{p^2 - M^2 - i\varepsilon}\; \exp\{-\,ip\cdot(y-x)\} \end{aligned}$$

is the adjoint Feynman propagator for the Dirac field.

3. We find

$$\begin{aligned} \left[\,F^{\rho\lambda}(x)\,,\,B(y)\,\right] \;&=\; \left[\,\partial^\rho_x A^\lambda(x) - \partial^\lambda_x A^\rho(x)\,,\,B(y)\,\right] \\ &=\; \partial^\rho_x\left[\,A^\lambda(x)\,,\,B(y)\,\right] - \partial^\lambda_x\left[\,A^\rho(x)\,,\,B(y)\,\right] \\ &=\; \partial^\rho_x(-\,i)\partial^\lambda_y D_0(\,y-x) + i\,\partial^\lambda_x\,\partial^\rho_y D_0(\,y-x) \\ &=\; -\,i\left(\partial^\rho_x\,\partial^\lambda_x - \partial^\lambda_x\,\partial^\rho_x\right) D_0(x-y) = 0 \end{aligned}$$

that also implies $\left[\,F^{\rho\lambda}(x)\,,\,B^{(-)}(y)\,\right] = 0$, so that the gauge invariant field strength Hermitean operators $F_{\mu\nu}(x) = F^\dagger_{\mu\nu}(x)$ appear to correspond to local observable operators in the covariant quantum field theory for the radiation field. Moreover we get

$$\begin{aligned} \left[\,F^{\mu\nu}(x)\,,\,F^{\rho\sigma}(y)\,\right] \;&=\; \left[\,\partial^\mu_x A^\nu(x) - \partial^\nu_x A^\mu(x)\,,\,\partial^\rho_y A^\sigma(y) - \partial^\sigma_y A^\rho(y)\,\right] \\ &=\; \partial^\mu_x\,\partial^\rho_y\left[\,A^\nu(x)\,,\,A^\sigma(y)\,\right] - \partial^\mu_x\,\partial^\sigma_y\left[\,A^\nu(x)\,,\,A^\rho(y)\,\right] \\ &-\; \partial^\nu_x\,\partial^\rho_y\left[\,A^\mu(x)\,,\,A^\sigma(y)\,\right] + \partial^\nu_x\,\partial^\sigma_y\left[\,A^\mu(x)\,,\,A^\rho(y)\,\right] \\ &=\; \left(\partial^\mu_x\,\partial^\rho_y\,g^{\nu\sigma} - \partial^\mu_x\,\partial^\sigma_y\,g^{\nu\rho} - \partial^\nu_x\,\partial^\rho_y\,g^{\mu\sigma} + \partial^\nu_x\,\partial^\sigma_y\,g^{\mu\rho}\right) \\ &\times\; iD_0(x-y) \end{aligned}$$

Thus we obtain

$$
\begin{aligned}
\left[\, F^{\imath 0}(x)\,,\, F^{\jmath 0}(y) \,\right] &= i\partial_0^2\,\delta_{\imath\jmath}\,D_0(x-y) - i\partial_\imath\,\partial_\jmath\,D_0(x-y) \\
&= (\,\triangle\,\delta_{\imath\jmath} - \partial_\imath\partial_\jmath\,)\,iD_0(x-y) \\
&\Rightarrow\ [\,E^\imath(x)\,,\,E^\jmath(y)\,] = 0 \qquad\quad \forall\,(x-y)^2<0 \\
\left[\, F^{\imath 0}(x)\,,\, F_{\jmath k}(y) \,\right] &= \left(\nabla_\jmath\,\delta^\imath_k - \nabla_k\,\delta^\imath_\jmath\right)\,\frac{i\partial}{\partial x_0}\triangle(x-y) \\
\left[\, E^\imath(x)\,,\, B^\ell(y) \,\right] &= \tfrac{1}{2}\,\varepsilon_{\jmath k \ell}\,[\,F_{\imath 0}(x)\,,\,F_{\jmath k}(y)\,] \\
&= \tfrac{1}{2}\,i\varepsilon_{\jmath k \ell}\,(\,\nabla_\jmath\,\delta_{\imath k} - \nabla_k\,\delta_{\imath\jmath}\,)\,\frac{\partial}{\partial x_0}\triangle(x-y)
\end{aligned}
$$

and from the equal time limit

$$
\lim_{x_0\to y_0}\frac{\partial}{\partial x_0}\,D_0(x-y) = \delta(\mathbf{x}-\mathbf{y})
$$

we obtain

$$
\begin{aligned}
&[\,E^\imath(x)\,,\,B^\imath(y)\,] = 0 \qquad\quad \forall\,(x-y)^2<0 \qquad\quad (\,\imath = 1,2,3\,) \\
&\lim_{x_0\to y_0}\ [\,E^\imath(x)\,,\,B^\jmath(y)\,] = i\varepsilon_{\imath\jmath k}\,\nabla_k\,\delta(\mathbf{x}-\mathbf{y})
\end{aligned}
$$

which shows that the electric and magnetic parts of the radiation field do not generally commute even at space-like separations. Finally we clearly get

$$
[\,B^\imath(x)\,,\,B^\jmath(y)\,] = 0 \qquad\quad \forall\,(x-y)^2<0
$$

QUANTUM FIELD THEORY **2.**

1. The so called non-Abelian field strength is an anti-symmetric matrix valued tensor $F_{\mu\nu}(x) = -F_{\nu\mu}(x)$ defined by

$$
\begin{aligned}
F_{\mu\nu}(x) &= F^a_{\mu\nu}(x)\,\boldsymbol{\tau}^a_F \equiv \frac{i}{g}\,[\,D_\mu\,,\,D_\nu\,] \\
&= \frac{i}{g}\left[\,\partial_\mu - ig\,A^a_\mu(x)\,\boldsymbol{\tau}^a_F\,, \partial_\nu - ig\,A^b_\nu(x)\,\boldsymbol{\tau}^b_F\,\right] \\
&= [\,\partial_\mu\,A^a_\nu(x) - \partial_\nu\,A^a_\mu(x)\,]\boldsymbol{\tau}^a_F - ig\,A^a_\mu(x)\,A^b_\nu(x)\,\left[\,\boldsymbol{\tau}^a_F\,,\,\boldsymbol{\tau}^b_F\,\right] \\
&= \boldsymbol{\tau}^a_F\left(\partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + g f^{abc} A^b_\mu(x)\,A^c_\nu(x)\right) \\
&= \partial_\mu\,A_\nu(x) - \partial_\nu\,A_\mu(x) - ig\,[\,A_\mu(x)\,,\,A_\nu(x)\,]
\end{aligned}
$$

and which evidently transforms in a homogeneous way

$$
F'_{\mu\nu}(x) = \frac{i}{g}\left[\,D'_\mu\,,\,D'_\nu\,\right] = U_\omega(x)\,F_{\mu\nu}(x)\,U^\dagger_\omega(x)
$$

This means that the field strengths tensor transforms according to the adjoint representation of the gauge group. In fact we find

$$\begin{aligned} F'_{\mu\nu}(x) &= \exp\{\, ig\,\omega^{a}(x)\boldsymbol{\tau}_F^{\,a}\}\, F_{\mu\nu}(x)\, \exp\{-\,ig\,\omega^{a}(x)\boldsymbol{\tau}_F^{\,a}\} \\ &= F_{\mu\nu}(x) + ig\,\omega^{a}(x)\, [\,\boldsymbol{\tau}_F^{\,a}\,,\, F_{\mu\nu}(x)\,] + O(\omega^2) \\ &= \Big(F^{\,c}_{\mu\nu}(x) - gf^{abc}\,\omega^{a}(x)\, F^{\,b}_{\mu\nu}(x)\; +\; O(\omega^2)\Big)\boldsymbol{\tau}_F^{\,c} \end{aligned}$$

that yields, after reshuffling of the group indexes,

$$\delta\, F^{\,a}_{\mu\nu}(x) = \, g\,\|\; I_a\;\|_{bc}\;\, F^{\,c}_{\mu\nu}(x)\,\omega^{\,b}(x)$$

where the well known property of the anti-Hermitean infinitesimal operators in the N^2-1 dimensional adjoint representation is employed, *viz.*

$$\|\; I_a\;\|_{bc} \equiv \; f^{acb} \qquad\qquad (\,a,b,c = 1,2,\ldots,N^2-1\,)$$

with $I_a + I_a^{\dagger} = 0$. Since the structure constants are real numbers for any Lie group, it turns out that the adjoint representation is always real. Hence, if we introduce the anti-symmetric tensor field strengths components

$$\{F^{\,a}_{\mu\nu}(x)\,|\, a = 1,2,\ldots,N^2-1\}$$

we can write its finite transformation law by raising the infinitesimal one to the exponential form, *viz.*,

$$\begin{aligned} \big(F^{\,a}_{\mu\nu}(x)\big)' &= \|\, \exp\{\,-\,gI_c\,\omega^{\,c}(x)\}\;\|_{ab}\;\, F^{\,b}_{\mu\nu}(x) \\ &= \Big(\delta^{ab} + gf^{cab}\,\omega^{\,c}(x) + \cdots\Big)F^{\,b}_{\mu\nu}(x) \\ &= F^{\,a}_{\mu\nu}(x) + gf^{abc}\, F^{\,b}_{\mu\nu}(x)\,\omega^{\,c}(x) + O(\omega^2) \end{aligned}$$

It is convenient to define the adjoint covariant derivative, *i.e.* the covariant derivative in the adjoint representation: namely,

$$\nabla_\mu = \partial_\mu + gA^c_\mu\, I_c \qquad\qquad \nabla^{\,ab}_\mu \;\equiv\; \partial_\mu\,\delta^{\,ab} - gf^{abc}\, A^c_\mu(x)$$

Then we obtain

$$\begin{aligned} \frac{1}{g}\,[\,\nabla_\mu\,,\,\nabla_\nu\,] &= \partial_\mu\, A^c_\nu(x)\, I_c - \partial_\nu\, A^c_\mu(x)\, I_c + g\, A^a_\mu(x)\, A^b_\nu(x)\,[\, I_a\,,\, I_b\,] \\ &= \Big(\partial_\mu\, A^c_\nu(x) - \partial_\nu\, A^c_\mu(x) + gf^{abc}\, A^a_\mu(x)\, A^b_\nu(x)\Big) I_c \\ &= \Big(\partial_\mu\, A^a_\nu(x) - \nabla^{\,ab}_\nu\, A^b_\mu(x)\Big) I_a = F^{\,a}_{\mu\nu}(x)\, I_a \end{aligned}$$

that keeps the same form as in the fundamental representation. If we rewrite the above commutator by exhibiting the group indexes we evidently obtain

$$\left[\,\nabla_\mu^{ab}\,,\,\nabla_\nu^{bc}\,\right] \;=\; g\,F_{\mu\nu}^{\,d}(x)\,\|\,I_d\,\|_{ac} = g\,f^{\,acd}\,F_{\mu\nu}^{\,d}(x)$$

It is also very useful to collect the infinitesimal form of the non-Abelian gauge transformations that reads, up to the first order in the small parameter functions $\delta\omega^a(x)$

$$\left\{\begin{array}{c} \delta\psi(x) = ig\,\delta\,\omega^{\,a}(x)\,\boldsymbol{\tau}_F^{\,a}\,\psi(x) \\ \delta A_\mu(x) = \nabla_\mu\,\delta\omega(x) = \boldsymbol{\tau}_F^{\,a}\,\delta A_\mu^a(x) = \boldsymbol{\tau}_F^{\,a}\,\nabla_\mu^{\,ab}\delta\omega^{\,b}(x) \\ \delta F_{\mu\nu}(x) = gf^{abc}\,F_{\mu\nu}^{\,b}(x)\,\delta\omega^{\,c}(x)\,\boldsymbol{\tau}_F^{\,a} \end{array}\right.$$

$$\left\{\begin{array}{c} \delta A_\mu^a(x) = \partial_\mu\delta\omega^{\,a}(x) - gf^{abc}\,\delta\omega^{\,b}(x)\,A_\mu^c(x) \\ \delta F_{\mu\nu}^{\,a}(x) = -\,gf^{abc}\,\delta\omega^{\,b}(x)\,F_{\mu\nu}^{\,c}(x) \end{array}\right.$$

2. According to the phenomenological Heisenberg IsoSpin Model, the two kinds of Nucleons can be supposed to be point-like and arranged into a doublet of Dirac fields

$$\Psi(x) = \begin{pmatrix} p(x) \\ n(x) \end{pmatrix}$$

transforming according to one of the fundamental representations of SU(2) that is called the Isotopic Spin or Isospin internal symmetry group, according to the original Heisenberg title to indicate this new quantum number. The point-like Nucleons are supposed to interact through the Yukava force carried by a spin-less Isoscalar real meson field $\pi^0(x)$, so that the classical Lagrangian of the present Heisenberg-Yukawa model model for nuclear matter reads

$$\begin{aligned} \mathcal{L} \;&=\; \Psi^\dagger(x)\gamma_0(i\partial\!\!\!/ - M)\,\Psi(x) + \tfrac{1}{2}\partial_\mu\pi^0(x)\partial^\mu\pi^0(x) - \tfrac{1}{2}m^2[\,\pi^0(x)\,]^2 \\ &-\; y\,\pi^0(x)\Psi^\dagger(x)\gamma_0\Psi(x) \qquad (\,y\in\mathbb{R}\,) \end{aligned}$$

The Lagrangian is invariant under the full Lorentz group, under the charge conjugation symmetry, the SU(2) isospin transformations on the spinor fields

$$\Psi(x) \quad\longmapsto\quad \Psi'(x) = \exp\left\{\tfrac{1}{2}\,i\sigma_a\,\theta_a\right\}\Psi(x) \qquad (\,a=1,2,3\,)$$

and the overall phase transformation on the SU(2) spinor doublet

$$\Psi(x) \quad\longmapsto\quad \Psi'(x) = \mathrm{e}^{\,i\varphi}\,\Psi(x)$$

where σ_a are the Pauli matrices while

$$0\le\theta<2\pi \qquad 0\le\varphi<2\pi \qquad \theta=\sqrt{\theta_1^2+\theta_2^2+\theta_3^2}$$

are the canonical coordinates of the internal symmetry group SU(2)×U(1). The invariance under the Abelian group of the phase transformations leads to conservation of the barion number B. Thus, if we measure the charge Q in units of the proton charge e, then we can write the relation $Q = T_3 + \frac{1}{2}B$ where

$$\begin{aligned} T_3 &= \int \mathrm{d}\mathbf{x}\, \Psi^{\dagger}(t,\mathbf{x})\, \tfrac{1}{2}\,\sigma_3 \Psi(t,\mathbf{x}) \\ &= \tfrac{1}{2}\int \mathrm{d}\mathbf{x}\, [\, p^{\dagger}(t,\mathbf{x})\, p(t,\mathbf{x}) - n^{\dagger}(t,\mathbf{x})\, n(t,\mathbf{x})\,] \\ B &= \int \mathrm{d}\mathbf{x}\, [\, p^{\dagger}(t,\mathbf{x})\, p(t,\mathbf{x}) + n^{\dagger}(t,\mathbf{x})\, n(t,\mathbf{x})\,] \\ \frac{Q}{e} &= \int \mathrm{d}\mathbf{x}\; p^{\dagger}(t,\mathbf{x})\, p(t,\mathbf{x}) \end{aligned}$$

The momentum space Feynman rules are the very same for both kinds of Nucleons as well as the Feynman rules for the incoming and outgoing particles and antiparticles: namely,

- scalar propagator: $D_F(k) = i[\,k^2 - m^2 + i\varepsilon\,]^{-1}$
- spinor propagator: $S^F_{\alpha\beta}(p) = i(\not{p} + M)_{\alpha\beta}\,(p^2 - M^2 + i\varepsilon)^{-1}$
- meson-Nucleon-Nucleon vertex: $-\,iy \qquad (\,p_1 + k - p_2 = 0\,)$
- for each loop of internal line labeled by ℓ : $\int \mathrm{d}^4\ell/(2\pi)^4$
- a factor $(-\,1)$ for each fermion loop
- incoming Nucleon: $u_{\,r}(p_1)\,,\ u_s(p_2)$
- outgoing Nucleon: $\bar{u}_{\,r\,'}(p_1')\,,\ \overline{u}_{s\,'}(p_2')$
- incoming anti-nucleon: $\bar{v}_{\,r}(p_1)\,,\ \overline{v}_s(p_2)$
- outgoing anti-nucleon: $v_{\,r\,'}(p_1')\,,\ v_{s\,'}(p_2')$

Let us now consider, for the sake of pedagogical simplicity, the pn collision for incident Nucleons momenta much below Mc, *i.e.* in the non-relativistic approximation. In such a circumstance, by comparing the amplitude for this process – up to the lowest order in the Yukawa coupling y – to the scattering amplitude of non-relativistic quantum mechanics in the Born approximation, we can extract the potential $V(r)$ created by the Yukawa field theory model.

As the two colliding Nucleons are distinguishable, only the diagram of Fig.1 does contribute to the lowest order y^2. Actually we understand the

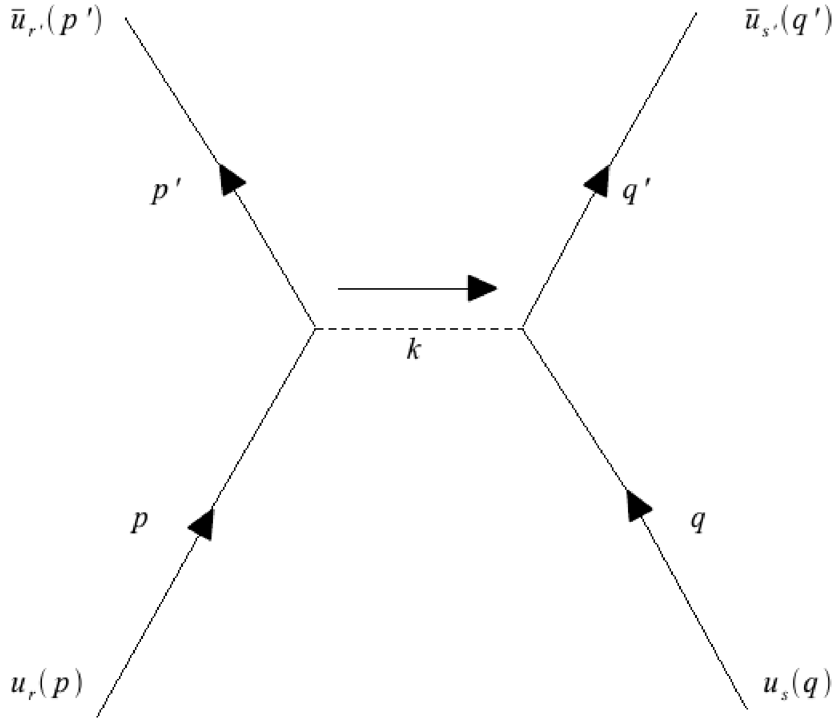

Figure 1: The lowest order diagram corresponding to Nucleon scattering in the Yukawa theory

incoming particles as free spinor particles of given energy momentum and polarization (p, r) and (q, s), while the outgoing free particles will carry the energy momentum and polarization labels (p', r') and (q', s') respectively. Hence, the application of the $S-$matrix basic formula to the Yukawa model with two kinds of spinor fields yields

$$\begin{aligned} S &= \mathbb{I} - iy \int \mathrm{d}x\, \pi^0_{\mathrm{int}}(x)[\,\overline{p}_{\mathrm{int}}(x)\, p_{\mathrm{int}}(x) + \overline{n}_{\mathrm{int}}(x)\, n_{\mathrm{int}}(x)\,] \\ &- y^2 \int \mathrm{d}x \int \mathrm{d}x'\, T\, \pi^0_{\mathrm{int}}(x)\, \overline{p}_{\mathrm{int}}(x)\, p_{\mathrm{int}}(x) \pi^0_{\mathrm{int}}(x')\, \overline{n}_{\mathrm{int}}(x')\, n_{\mathrm{int}}(x') \\ &+ \cdots\cdots \end{aligned}$$

and the further application of the Wick's theorem to this scattering operator matrix element gives rise to only a single non-vanishing term, *viz.,*

$$\begin{aligned} & (-iy)^2 \langle\, 0\,|\, c_{\,r'}(\,p'\,)\, C_{\,s'}(\,q'\,) : \bar{n}^{\,(+)}_{\,x'}\, n^{\,(-)}_{\,x'}\, \overline{p}^{\,(+)}_{\,x}\, p^{\,(-)}_{\,x} : C^{\,\dagger}_{\,s}(\,q)\, c^{\,\dagger}_{\,r}(\,p)\,|\,0\,\rangle\, D_{xx'} \\ = \; & (-1)(-iy)^2 \langle\, 0\,|\, c_{\,r'}(\,p'\,)\, C_{\,s'}(\,q'\,)\, \bar{n}^{\,(+)}_{\,x'}\, \underbrace{n^{\,(-)}_{\,x'}\, c^{\,\dagger}_{\,r}(\,p)}\, \overline{p}^{\,(+)}_{\,x}\, \underbrace{p^{\,(-)}_{\,x}\, C^{\,\dagger}_{\,s}(\,q)}\,|\,0\,\rangle\, D_{xx'} \\ = \; & (+1)\,(-iy)^2\, \underbrace{c_{\,r'}(\,p'\,)\, \bar{n}_{\,x'}}\; \underbrace{n_{\,x'}\, c^{\,\dagger}_{\,r}(\,p)}\; D_{xx'}\; \underbrace{C_{\,s'}(\,q'\,)\, \overline{p}_{\,x}}\; \underbrace{p_{\,x}\, C^{\,\dagger}_{\,s}(\,q)} \\ \Rightarrow \; & \bar{u}_{\,r'}(\,p'\,)\, u_{\,r}(\,p)\; \frac{-\,iy^2}{(\,p' - p)^2 - m^2}\; \overline{u}_{\,s'}(\,q'\,)\, u_{\,s}(\,q) \qquad (\,p + q = p' + q'\,) \end{aligned}$$

where we have indicated with small and capital letters the creation and destruction operators of the neutron and proton particles respectively. Here the Dirac bispinor indexes have been always understood, to avoid too heavy notations while, of course, we have

$$r, s, r', s' = 1, 2\,,\;\; p^{\,2} = q^{\,2} = p'^{\,2} = q'^{\,2} = M^{\,2}$$

Notice that we have employed the same form of the spin states for both Nucleons, which is true in the equal masses approximation.

3. The perturbative expansion of the collision matrix in QED with two kinds of spinor field carrying equal electric charges though different masses reads

$$\begin{aligned} S &= \mathbb{I} + ie \int \mathrm{d}y\, A^{\mu}_{\mathrm{int}}(y) \overline{\psi}_{\,\mathrm{int}}(y) \gamma_\mu \psi_{\,\mathrm{int}}(y) \\ &+ ie \int \mathrm{d}x\, A^{\nu}_{\mathrm{int}}(x) \overline{\Psi}_{\,\mathrm{int}}(x) \gamma_\nu \Psi_{\,\mathrm{int}}(x) \\ &- e^2 \int \mathrm{d}x \int \mathrm{d}y\, T A^{\mu}_{\mathrm{int}}(x) \overline{\psi}_{\,\mathrm{int}}(x) \gamma_\mu \psi_{\,\mathrm{int}}(x) A^{\nu}_{\mathrm{int}}(y) \overline{\Psi}_{\,\mathrm{int}}(y) \gamma_\nu \Psi_{\,\mathrm{int}}(y) \\ &+ \cdots\cdots \end{aligned}$$

To the first order in $e/\sqrt{2hc}$ some processes might occur in perturbation theory, in which three physical particles – one photon and two Dirac particles – would appear in the initial and final states on the mass shells. It can be readily seen, however, that those kinds of processes are impossible, owing to energy momentum conservation. If we denote by k^μ the photon momentum and by p^ν, q^ρ the Dirac particles momenta respectively, then the energy-momentum conservation is expressed by the equality $k = p \pm q$, the sign plus being related to a particle-antiparticle pair, the minus sign being instead referred to a 2-particles or a 2-antiparticles pair. The above equality is in fact impossible because $k^2 = 0$ while for *e.g.* $\mathbf{q} = 0$ we get

$$(\,p \pm q\,)^2 = 2(M^2 \pm p \cdot q) = 2(M^2 \pm p_0 q_0 \mp \mathbf{p} \cdot \mathbf{q}) = 2M(M \pm p_0)$$

and since $p_0 > M$ we find either $(\,p + q\,)^2 > 0$ or $(\,p - q\,)^2 < 0$. Hence, the first nontrivial term in the collision matrix becomes

$$\begin{aligned} S &= -e^2 \int \mathrm{d}x \int \mathrm{d}y \left(T\, A^\mu_{\mathrm{int}}(x)\, A^\nu_{\mathrm{int}}(y) \right) \\ &\times \left(T\, \overline{\psi}_{\mathrm{int}}(x) \gamma_\mu \psi_{\mathrm{int}}(x) \overline{\Psi}_{\mathrm{int}}(y) \gamma_\nu \Psi_{\mathrm{int}}(y) \right) + \cdots\cdots \end{aligned}$$

owing to the commutation between photon and Dirac field operators. Thus, for $e^-\mu^-$ elastic scattering, the Wick's theorem yields, up to the leading order and in natural units,

$$\begin{aligned} & -e^2\, \langle 0\,|\, c_s(q)\, C_{s'}(\,q'\,) : \bar{\psi}^{(+)}_x\, \gamma_\mu\, \psi^{(-)}_x\, D^{\mu\nu}_{xy}\, \overline{\Psi}^{(+)}_y\, \gamma_\nu\, \Psi^{(-)}_y : C^\dagger_{r'}(\,p'\,)\, c^\dagger_r(p)\,|\,0\,\rangle \\ = \quad & -e^2\, \langle 0\,|\, c_s(q)\, \bar{\psi}^{(+)}_x\, \gamma_\mu\, \psi^{(-)}_x\, \underbrace{C_{s'}(\,q'\,)\, \overline{\Psi}^{(+)}_y}\, \gamma_\nu\, \underbrace{\Psi^{(-)}_y\, C^\dagger_{r'}(\,p'\,)}\, c^\dagger_r(p)\,|\,0\,\rangle\, D^{\mu\nu}_{xy} \\ = \quad & -e^2\, \langle 0\,|\, \underbrace{c_s(\,q)\, \bar{\psi}^{(+)}_x}\, \gamma_\mu\, \underbrace{\psi^{(-)}_x\, c^\dagger_r(p)}\, D^{\mu\nu}_{xy}\, \underbrace{C_{s'}(\,q'\,)\, \overline{\Psi}^{(+)}_y}\, \gamma_\nu\, \underbrace{\Psi^{(-)}_y\, C^\dagger_{r'}(\,p'\,)}\,|\,0\,\rangle \\ \Rightarrow \quad & -e^2\, \bar{u}_s(\,q)\, \gamma^\mu\, u_r(p)\, \frac{-i}{(\,p - p'\,)^2}\, \overline{U}_{s'}(\,q'\,)\, \gamma_\mu\, U_{r'}(\,p'\,) \qquad (\,p + p' = q + q'\,) \end{aligned}$$

where, of course, we have

$$r, s, r', s' = 1, 2\,,\; p^2 = q^2 = p'^2 = q'^2 = M^2$$

Putting all pieces together we find from the Feynman graph the dimensionless transition amplitude

$$\bar{u}_s(\,q)\, ie\, \gamma^\mu\, u_r(\,p)\, \frac{-i}{(\,p - q)^2}\, \overline{U}_{s'}(\,q'\,)\, ie\, \gamma_\mu\, U_{r'}\,(p'\,)$$

Notice that the Feynman gauge photon propagator, which represents the electromagnetic interaction in the present lowest order amplitude, can be suitably rewritten in the very suggestive form

$$D_{\mu\nu}(k) = \frac{g_{\mu\nu}}{it}$$

where t is the Mandelstam variable which corresponds to the invariant norm of the transferred 4-momentum $k = p - q = q' - p'$, *viz.*,

$$t \equiv \hbar^2 k^2 = \hbar^2 (p-q)^2 = 2M^2c^2 \left[1 - \sqrt{1 + \frac{\hbar^2 \mathbf{q}^2}{M^2c^2}} \right] < 0$$

Here the rest frame of the incoming particle $p = (Mc, 0, 0, 0)$ has been used, without loss of generality thanks to the Lorentz invariance, to exhibit the space-like nature of the transferred momentum t. This means in turn that the exchanged photon, which mediates the electromagnetic interaction between the two charged Dirac spinor particles, is virtual and space-like, *i.e.* off its mass shell $k^2 = 0$, and that all the four kinds of polarization, physical and non-physical, do indeed carry the Coulomb interaction in the manifestly covariant Feynman gauge. In the non-relativistic limit we can approximate as follows:

$$\hbar p \approx (Mc, \hbar\mathbf{p}) \qquad \hbar q \approx (Mc, \hbar\mathbf{q})$$

$$\hbar p' \approx (Mc, \hbar\mathbf{p}') \qquad \hbar q' \approx (Mc, \hbar\mathbf{q}')$$

$$(p - p')^2 \approx -|\mathbf{p} - \mathbf{p}'|^2$$

$$u_1(p) \approx \sqrt{\frac{Mc}{\hbar}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u_2(p) \approx \sqrt{\frac{Mc}{\hbar}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

so that

$$u_s^\dagger(q)\, u_r(p) \approx 2\,\frac{Mc}{\hbar}\,\delta_{rs} \qquad u_{s'}^\dagger(q')\, u_{r'}(q') \approx 2\,\frac{Mc}{\hbar}\,\delta_{r's'}$$

$$\bar{u}_s(q)\,\gamma^k\, u_r(p) = u_s^\dagger(q)\,\alpha^k\, u_r(p) \approx 0$$

$$\bar{u}_{s'}(q')\,\gamma_k\, u_{r'}(p') = u_{s'}^\dagger(q')\,\alpha_k\, u_{r'}(p') \approx 0$$

for $r, r', s, s' = 1, 2$, in such a manner that the particle spin is conserved in the non-relativistic regime. Then we eventually come to the non-relativistic approximation

$$\frac{-ie^2c^2}{\hbar^2|\mathbf{p} - \mathbf{q}|^2}\, 2M\,\delta_{rs}\, 2M\,\delta_{r's'} = 4\pi i\, T_{\mathbf{p},\mathbf{q}}\, 2\,\frac{Mc}{\hbar}\,\delta_{rs}\,\delta_{r's'}$$

and consequently

$$T_{\mathbf{p},\mathbf{q}} = f(\theta) = -\,\frac{2\alpha Mc}{\hbar\,|\mathbf{p} - \mathbf{q}|^2}$$

which corresponds to the repulsive Coulomb potential

$$V(r) = \frac{e^2}{4\pi r} = \frac{\alpha}{r} \qquad \widetilde{V}(|\mathbf{p}-\mathbf{q}|) = \frac{e^2}{|\mathbf{p}-\mathbf{q}|^2}$$

so that

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right) = \frac{M^2\alpha^2c^2}{4\hbar^2\,|\mathbf{p}|^4\,\sin^4(\theta/2)} = \frac{\alpha^2\hbar^2c^2}{16E^2\,\sin^4(\theta/2)} \qquad (\hbar^2\mathbf{p}^2 = 2ME)$$

which is nothing but the celebrated Rutherford exact cross-section. For an antiparticle-particle scattering we have to make the replacement

$$\bar{u}_s(q)\,ie\gamma^\mu\,u_r(p) \quad \leftrightarrow \quad \bar{v}_r(p)\,(-ie\gamma^\mu)\,v_s(q)$$

and owing to

$$\bar{v}_s(q)\,\gamma^0\,v_r(p) \;\approx\; 2M\,\delta_{rs} \qquad \text{et cetera}$$

the sign of the non-relativistic Coulomb potential is opposite as it does. As a final remark we discuss about gauge invariance. One is always free to replace the photon propagator in the Feynman gauge with the the most general expression in a Lorentz invariant non-homogeneous Lorenz gauge $\partial_\mu A^\mu(x) = \xi B(x)$ that yields

$$\widetilde{D}^{c}_{\lambda\mu}(k\,;\xi) = \frac{i\hbar c}{k^2+i\varepsilon}\left(-g_{\lambda\mu} + \frac{1-\xi}{k^2+i\varepsilon}\,k_\lambda\,k_\mu\right)$$

with $k = p - p' = q' - q$. Now, if we recall the Dirac equations for the spin states, *viz.*,

$$(\not{p} - M)u_r(p) = 0 = \bar{u}_{r'}(p')(\not{p}' - M)$$
$$(\not{q} - M)u_s(q) = 0 = \bar{u}_{s'}(q')(\not{q}' - M)$$

then we obtain the matrix general element

$$\begin{aligned} & \bar{u}_s(q)\,ie\gamma^\mu\,u_r(p)\,\frac{-i}{(p-q)^2}\,\bar{u}_{s'}(q')\,ie\gamma^\nu\,u_{r'}(p') \\ \times & \left[g_{\mu\nu} - \frac{1-\xi}{(p-q)^2+i\varepsilon}\,(p-q)_\mu\,(q'-p')_\nu\right] \\ = & \;\bar{u}_s(q)\,\gamma^\mu\,u_r(p)\,\frac{ie^2}{(p-q)^2}\,\bar{u}_{s'}(q')\,\gamma_\mu\,u_{r'}(p') \end{aligned}$$

which endorses gauge invariance, *i.e.* $\xi-$independence, of the lowest order scattering amplitude. However, it turns out that this fundamental feature holds true to any order, what corresponds to the so called Ward's identity.

0.3 *Wednesday 17 April 2019*
1. Basic QFT Medley

FIELD THEORY **1.**

Basic Quantum Field Theory Medley

1. Write in explicit form the generic element of SL(2,$\mathbb{C}$) as a function of the canonical coordinates $(\boldsymbol{\alpha}, \boldsymbol{\eta})$ of the proper Lorentz group.

2. Show that for a classical Action which is invariant under space-time translations the canonical energy-momentum tensor does satisfy the continuity equation $\partial_\mu T^\mu{}_\nu(x) = 0$.

3. Consider a Klein-Gordon scalar quantum field $\phi(t, \mathbf{x})$. Write its normal modes expansion. Verify that for $\omega_{\mathbf{k}} = c\sqrt{\mathbf{k}^2 + m^2c^2/\hbar^2}$

$$cP_0 = H = \int \mathrm{d}\mathbf{x} : T_{00}(t, \mathbf{x}) : = \int \mathrm{d}\mathbf{k}\, \hbar\omega_{\mathbf{k}}\, a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$$

4. Show that the bi-linear expression $\overline{\psi}(x)\gamma_5\psi(x)$ is a pseudo-scalar, where $\psi(x)$ is a classical four components Dirac spinor field, while the further expression $\overline{\psi}(x)\gamma^\mu\gamma_5\psi(x)$ is a pseudo-vector.

5. Write the normal modes expansion of the quantum radiation field in the Feynman gauge and list the canonical commutation relations among the creation-destruction operators.

0.4 *Thursday 19 February 2019*
1. Field Theory Medley 2. $\pi^0 N$ Elastic Collision

FIELD THEORY **1.**

1. **Poincaré Invariance of the Klein-Gordon Quantum Field.** The operator valued tempered distribution which describe a Klein-Gordon quantum scalar field is provided by the normal modes expansion

$$\phi(x) = \int \mathrm{D}k \left[a(k)\, e^{-ik\cdot x} + a^\dagger(k)\, e^{ik\cdot x} \right]_{k_0 = \omega_{\mathbf{k}}}$$

where $\int \mathrm{D}k \equiv \int \mathrm{d}\mathbf{k}/(2\pi)^3\, 2\,\omega_{\mathbf{k}}$ while the creation-destruction operator satisfy the canonical commutation relations

$$[\,a(k)\,,\,a(k')\,] = 0 \qquad [\,a(k)\,,\,a^\dagger(k')\,] = (2\pi)^3\, 2k_0\,\delta(\mathbf{k}-\mathbf{k}') \quad (\,k_0 = \omega_{\mathbf{k}}\,)$$

(i) Write the unitary operator of the Poincaré transformations on the Fock space of states of a quantum Klein-Gordon field.

(ii) Evaluate its action upon the creation-destruction operators.

(iii) Verify that the Poincaré transformations look the same for classical and quantum fields.

2. Charge conjugation of a Dirac field is defined to be $\psi^c(x) = \gamma^2\,\psi^*(x)$. Obtain the charge conjugation for the following expressions

$$\overline{\psi} i \partial\!\!\!/ \psi, \overline{\psi}\psi, \overline{\psi}\gamma_5\psi, \overline{\psi}\gamma^\mu\psi, \overline{\psi}\gamma^\mu\gamma_5\psi$$

3. Consider the quantum radiation field in the absence of charged matter. Write the quantum counterpart of the Maxwell equations. Evaluate the commutator $[\,\mathbf{B}(x), \mathbf{B}(y)\,]$ where $\mathbf{B} = \nabla \times \mathbf{A}$.

FIELD THEORY **2.**

The two kinds of Nucleons can be arranged into a doublet of Dirac fields

$$\Psi(x) = \left(\begin{array}{c} p(x) \\ n(x) \end{array} \right)$$

transforming according to one of the fundamental representations of SU(2) that is called the Isospin internal symmetry group. The Nucleons interact through the Yukava force carried by a spin-less Iso-scalar field $\pi(x)$, the 1-particle states of which are the neutral π^0 mesons, so that the classical Lagrangian reads

$$\mathcal{L} = \overline{\Psi}(x)\,(i\partial\!\!\!/ - M)\,\Psi(x) + \tfrac{1}{2}\partial_\mu\pi(x)\partial^\mu\pi(x) - \tfrac{1}{2}m^2\pi^2(x) - g\pi(x)\overline{\Psi}(x)\,\Psi(x)$$

where $g > 0$ is the Yukawa coupling while we safely assume[1] $m_p \approx m_n \approx M$.

1. Find all the symmetries of the Action
2. Write the Feynman rules in momentum space
3. Find the lowest order non-polarized differential cross section for the $\pi^0 N$ elastic scattering. Suppose the Nucleon at rest before the collision and the π^0 momenta, before and after the collision, large enough to disregard the meson rest mass.

[1] Experimentally one finds $(m_n - m_p)/(m_n + m_p) \simeq 0.7 \times 10^{-3}$, while $m_{\pi^0}/m_p \simeq 0.14$.

Solution.

FIELD THEORY **1.**

1. It turns out that to each element of the restricted Poincaré group, which is uniquely specified by the ten canonical coordinates

$$\left(\omega^{\mu\nu}, \mathrm{a}^{\lambda}\right) = \left(\boldsymbol{\alpha}, \boldsymbol{\eta}, \mathrm{a}^{\lambda}\right) = (\omega, \mathrm{a})$$

with $|\,\boldsymbol{\alpha}\,| < 2\pi$, $\boldsymbol{\eta} \in \mathbb{R}^3$, $\mathrm{a}^{\lambda} \in \mathbb{R}^4$, there will correspond a unitary operator such that

$$U(\omega, \mathrm{a})\,|\,0\,\rangle = \exp\left\{\frac{i}{\hbar}\,\mathrm{a}^{\mu}\,P_{\mu} - \frac{i}{2\hbar}\,\omega^{\,\rho\sigma}L_{\rho\sigma}\right\}|\,0\,\rangle = |\,0\,\rangle$$

$$\langle\,0\,|\,U^{\,\dagger}(\omega, \mathrm{a}) = \langle\,0\,|\,\exp\left\{-\frac{i}{\hbar}\mathrm{a}^{\mu}\,P_{\mu} + \frac{i}{2\hbar}\,\omega^{\,\rho\sigma}L_{\rho\sigma}\right\} = \langle\,0\,|$$

which means that the vacuum state is Poincaré invariant or, in other words, that $IO(1,3)$ acts trivially on the one dimensional ray of the Fock space generated by the vacuum state for a Klein-Gordon scalar quantum field. In the case of the Klein-Gordon neutral field, the explicit form for the Hermitean generators is provided by the normal ordered expressions

$$\begin{aligned}
P_0 &= \int \mathrm{d}\mathbf{x}\;\tfrac{1}{2} : \Pi^{2}\,(x) + \nabla\phi(x)\cdot\nabla\phi(x) + m^2\phi^2(x) : = \sum_{\mathbf{k}}\;\omega_{\mathbf{k}}\,a^{\dagger}_{\mathbf{k}}\,a_{\mathbf{k}} \\
P_k &= \int \mathrm{d}\mathbf{x}\;\; : \Pi(x)\,\partial_k\phi(x) : = \sum_{\mathbf{k}}\;\mathbf{k}\,a^{\dagger}_{\mathbf{k}}\,a_{\mathbf{k}} \\
L_{\imath\jmath} &= \int \mathrm{d}\mathbf{x}\;\; : x_{\imath}\,\Pi(x)\,\partial_{\jmath}\phi(x) - x_{\jmath}\,\Pi(x)\,\partial_{\imath}\phi(x) : \\
&= \sum_{\mathbf{k}}\;\frac{i}{2}\left(k_{\,\imath}\;a^{\dagger}_{\mathbf{k}}\,\frac{\overleftrightarrow{\partial}}{\partial k^{\,\jmath}}\,a_{\mathbf{k}} \;-\; k_{\,\jmath}\;a^{\dagger}_{\mathbf{k}}\,\frac{\overleftrightarrow{\partial}}{\partial k^{\,\imath}}\,a_{\mathbf{k}}\right) \\
L_{\,0k} &= x_0\,P_k - m\,X_k(t) = \frac{i}{2}\sum_{\mathbf{p}}\omega_{\mathbf{p}}\;a^{\dagger}_{\mathbf{p}}\,\frac{\overleftrightarrow{\partial}}{\partial\,p^{k}}\,a_{\mathbf{p}} \\
X^k(t) &= \frac{1}{2m}\int \mathrm{d}\mathbf{x}\,x^{\,k} : \Pi^{2}(x) + \nabla\phi(x)\cdot\nabla\phi(x) + m^2\phi^{2}\,(x) :
\end{aligned}$$

Moreover, it can be actually verified that the creation-destruction operators undergo the following changes under a Poincaré transformation: namely,

$$\begin{aligned}
&a'(k) \equiv U(\omega, \mathrm{a})\,a(k)\,U^{\,-1}(\omega, \mathrm{a}) \;=\; a(k')\,\exp\{-\,i\,k\cdot\mathrm{a}\} \\
&k'_{\mu} = \Lambda_{\mu}^{\;\;\nu}\,k_{\nu} \qquad k_0 = \omega_{\mathbf{k}} \qquad g^{\mu\nu}\,k'_{\mu}k'_{\nu} = k'^{\,2} = k^2 = m^2 \\
&a'^{\dagger}(k) \;\equiv\; U(\omega, \mathrm{a})\,a^{\dagger}(k)\,U^{\,-1}(\omega, \mathrm{a}) \;=\; \exp\{\,i\,k\cdot\mathrm{a}\}\,a^{\dagger}(k')
\end{aligned}$$

which endorses the Poincaré invariance, up to a phase factor, of the creation and annihilation operators of the Klein-Gordon quantum scalar field.

As a matter of fact, consider an infinitesimal Poincaré transformation

$$U(\delta\omega,\delta\mathrm{a})\,a(k)\,U^{\,\dagger}(\delta\omega,\delta\mathrm{a}) \simeq a(k) + i\left[\,\delta\mathrm{a}^{\,\mu}\,P_\mu - \tfrac12\,\delta\omega^{\,\rho\sigma}L_{\rho\sigma}\,,\,a(k)\,\right]$$

From the canonical commutation relations it is straightforward to show that

$$\begin{aligned}
[\,a(k)\,,\,P_\mu\,] &= [\,(\,u_{\,k}\,,\,\phi\,)\,,\,P_\mu\,] = i\,(\,u_{\,k}\,,\,\partial_\mu\,\phi\,)\\
&= \int \mathrm{D}p\;p_\mu\left[\,a(p)\;(\,u_{\,k}\,,\,u_{\,p}) - a^{\,\dagger}(p)\;\left(\,u_{\,k}\,,\,u_{\,p}^{*}\right)\,\right] = k_\mu\,a(k)
\end{aligned}$$

Moreover we have

$$\begin{aligned}
[\,a(k)\,,\,L_{\mu\nu}\,] &= [\,(\,u_{\,k}\,,\,\phi\,)\,,\,L_{\mu\nu}\,] = \int \mathrm{d}\mathbf{x}\;u_{\,k}^{*}(t,\mathbf{x})\,i\overleftrightarrow{\partial}_0[\,\phi(t,\mathbf{x})\,,\,L_{\mu\nu}\,]\\
&= -\int \mathrm{d}\mathbf{x}\;u_{\,k}^{*}(t,\mathbf{x})\,\overleftrightarrow{\partial}_0\Big(x_\mu\,\partial_\nu\,\phi(t,\mathbf{x}) - x_\nu\,\partial_\mu\,\phi(t,\mathbf{x})\Big)\\
&= \int \mathrm{D}p\;p_\nu\left[\,a^{\dagger}(p)\,\frac{i\partial}{\partial\,p^{\,\mu}}\;\left(\,u_{\,k}\,,\,u_{\,p}^{*}\right) + a(p)\,\frac{i\partial}{\partial\,p^{\,\mu}}\,(\,u_{\,k}\,,\,u_{\,p})\,\right] - \mu\,\leftrightarrow\,\nu\\
&= -\int \mathrm{D}p\;p_\nu\left[\,a^{\,\dagger}(p)\,\frac{i\partial}{\partial\,k^{\,\mu}}\;\left(\,u_{\,k}\,,\,u_{\,p}^{*}\right) + a(p)\,\frac{i\partial}{\partial\,k^{\,\mu}}\,(\,u_{\,k}\,,\,u_{\,p})\,\right] - \mu\,\leftrightarrow\,\nu\\
&= -\,\frac{i\partial}{\partial\,k^{\,\mu}}\int \mathrm{D}p\;p_\nu\,a(p)\;(\,u_{\,k}\,,\,u_{\,p}) - \mu\,\leftrightarrow\,\nu\\
&= -\,\frac{i\partial}{\partial\,k^{\,\mu}}\,(k_\nu\,a(k)) - \mu\,\leftrightarrow\,\nu = i\,k_\mu\,\frac{\partial}{\partial\,k^{\,\nu}}\,a(k) - i\,k_\nu\,\frac{\partial}{\partial\,k^{\,\mu}}\,a(k)
\end{aligned}$$

where use has been made of the inversion formulæ. Hence, under an infinitesimal Poincaré transformation we get

$$\begin{aligned}
&U(\delta\omega,\delta\mathrm{a})\,a(k)\,U^{\,-1}(\delta\omega,\delta\mathrm{a}) \simeq a(k) - i\left[\,a(k)\,,\,\delta\mathrm{a}^{\,\mu}\,P_\mu - \tfrac12\,\delta\omega^{\,\rho\sigma}L_{\rho\sigma}\,\right]\\
&= \left\{1 - i\,\delta\mathrm{a}\cdot k - \tfrac12\,\delta\omega^{\,\mu\nu}\left(k_\mu\,\frac{\partial}{\partial\,k^{\,\nu}} - k_\nu\,\frac{\partial}{\partial\,k^{\,\mu}}\right)\right\}a(k)
\end{aligned}$$

so that we eventually find

$$a^{\,\prime}(k) - a(k)\;\simeq\;\delta a(k)\;\simeq\;\left\{\tfrac12\,\epsilon^{\,\mu\nu}\left(k_\nu\,\frac{\partial}{\partial\,k^{\,\mu}} - k_\mu\,\frac{\partial}{\partial\,k^{\,\nu}}\right) - i\,k_\mu\,\epsilon^{\,\mu}\right\}a(k)$$

where we have identified as customary $\delta\mathrm{a}^{\,\mu} \equiv \epsilon^{\,\mu}\,,\ \delta\omega^{\,\mu\nu} \equiv \epsilon^{\,\mu\nu}$. Moreover, the action of an infinitesimal Lorentz transformation on the wave tetra-vector yields

$$a(\Lambda\,k) - a(k)\;\simeq\;a(k+\delta k) - a(k) = \delta k^{\,\mu}\,\frac{\partial}{\partial\,k^{\,\mu}}\,a(k) = \epsilon^{\,\mu\nu}\,k_\nu\,\frac{\partial}{\partial\,k^{\,\mu}}\,a(k)$$

in such a manner that we can finally get the finite transformation rule

$$\begin{aligned}
U(\omega,\mathrm{a})\,a(k)\,U^{\,-1}(\omega,\mathrm{a}) &= \exp\{-\,i\,k\cdot\mathrm{a}\}\,a(\Lambda\,k)\\
&\simeq (1 - i\,k_\mu\,\mathrm{a}^{\,\mu} + \cdots)\left(1 - \omega^{\,\mu\nu}\,k_\nu\,\frac{\partial}{\partial\,k^{\,\mu}} + \cdots\right)a(k)\\
&= \left\{1 - i\,k_\mu\,\mathrm{a}^{\,\mu}\,a(k) + \tfrac12\,\omega^{\,\mu\nu}\left(k_\nu\,\frac{\partial}{\partial\,k^{\,\mu}} - k_\mu\,\frac{\partial}{\partial\,k^{\,\nu}}\right)\right\}a(k) + \cdots
\end{aligned}$$

and consequently

$$a'(k) \equiv U(\omega,\mathrm{a})\,a(k)\,U^{\dagger}(\omega,\mathrm{a}) \;=\; a(\Lambda\,k)\exp\{-\,i\,k\cdot\mathrm{a}\} \qquad k_0=\omega_{\mathbf{k}}$$

as claimed. It is worthwhile to notice that by repeating the very same steps for the inverse Poincaré unitary similarity transformation we obtain

$$a'(k) \equiv U^{-1}(\omega,\mathrm{a})\,a(k)\,U(\omega,\mathrm{a}) \;=\; a(\Lambda^{-1}k)\exp\{\,i\,k\cdot\mathrm{a}\} \qquad (\,k_0=\omega_{\mathbf{k}}\,)$$

with

$$U^{-1}(\omega,\mathrm{a}) = U(-\,\omega,-\,\mathrm{a}) = U^{\dagger}(\omega,\mathrm{a})$$

which implies in turn

$$\begin{aligned} a'(k'\,) &\equiv U^{\dagger}(\omega,\mathrm{a})\,a(\Lambda\,k)\,U(\omega,\mathrm{a}) \\ &= a(k)\exp\{\,i\,k\cdot\mathrm{a}\} \qquad\qquad (\,k_0=\omega_{\mathbf{k}}\,) \end{aligned}$$

showing that for any homogeneous Lorentz transformation the creation and annihilation operators are invariant. From the transformation law of the creation-destruction operators we immediately obtain the following identity between operator valued tempered distributions, which looks the very same as that one for the classical scalar field, *viz.,*

$$\begin{aligned} \phi'(x') &\equiv U^{-1}(\omega,\mathrm{a})\,\phi(x')\,U(\omega,\mathrm{a}) \\ &= \int \mathrm{D}k'\,\Big[\,a'(k')\,\exp\{-i\,k'\cdot x'\} + \mathrm{h.\,c.}\Big]_{k'_0=\omega_{\mathbf{k}'}} \\ &= \int \mathrm{D}(\Lambda k)\,\Big[\,e^{\,ik\cdot\mathrm{a}}\,a(k)\,e^{-\,ik\cdot(x+\mathrm{a})} + \mathrm{h.\,c.}\Big]_{k_0=\omega_{\mathbf{k}}} \;=\; \phi(x) \end{aligned}$$

Concerning the discrete symmetry transforms, parity and time reversal, we find instead

$$\begin{aligned} \phi'(x') &= \mathcal{P}\phi(x')\mathcal{P}^{-1} = \mathcal{P}\phi(t,-\,\mathbf{x})\mathcal{P}^{\dagger} \\ &= \int \mathrm{d}\mathbf{k}\;a_{-\,\mathbf{k}}\,u_{\mathbf{k}}(t,-\,\mathbf{x}) \;+\; \mathrm{H.c.} \\ &= \int \mathrm{d}\mathbf{k}\;a_{-\,\mathbf{k}}\,[\,(2\pi)^3\,2\omega_{\mathbf{k}}\,]^{-\,\frac12}\exp\{-\,it\omega_{\mathbf{k}} - i\,\mathbf{k}\cdot\mathbf{x}\,\} \;+\; \mathrm{H.c.} \\ &= \int \mathrm{d}\mathbf{k}\;a_{\mathbf{k}}\,[\,(2\pi)^3\,2\omega_{\mathbf{k}}\,]^{-\,\frac12}\exp\{-\,it\omega_{\mathbf{k}} + i\,\mathbf{k}\cdot\mathbf{x}\,\} \;+\; \mathrm{H.c.} \\ &= \phi(x) \end{aligned}$$

with $\mathcal{P}=\mathcal{P}^{-1}=\mathcal{P}^{\dagger}$ ($\mathcal{P}^{2}=\mathbb{I}$) unitary and self-adjoint operator on the Fock space, while

$$\phi'(x') \;=\; \mathcal{T}\,\phi(x')\,\mathcal{T}^{-1} = \mathcal{T}\,\phi(-\,t,\mathbf{x})\,\mathcal{T}^{\dagger}$$

$$
\begin{aligned}
&= \int \mathrm{d}\mathbf{k}\; a_{-\mathbf{k}}\, u^*_{\mathbf{k}}(-t,\mathbf{x}) \;+\; \text{H.c.} \\
&= \int \mathrm{d}\mathbf{k}\; a_{-\mathbf{k}}\,[\,(2\pi)^3\, 2\omega_{\mathbf{k}}\,]^{-\frac{1}{2}} \exp\{-\,it\omega_{\mathbf{k}} - i\,\mathbf{k}\cdot\mathbf{x}\,\} \;+\; \text{H.c.} \\
&= \int \mathrm{d}\mathbf{k}\; a_{\mathbf{k}}\,[\,(2\pi)^3\, 2\omega_{\mathbf{k}}\,]^{-\frac{1}{2}} \exp\{-\,it\omega_{\mathbf{k}} + i\,\mathbf{k}\cdot\mathbf{x}\,\} \;+\; \text{H.c.} \\
&= \phi(x)
\end{aligned}
$$

where use has been made of the anti-linear and anti-unitary property of the time reversal operator $\forall\,\alpha,\beta\in\mathbb{C}\,\vee\,|\,u\,\rangle\,,\,|\,v\,\rangle\in\mathcal{F}$

$$
\mathcal{T}\,(\alpha\,|\,u\,\rangle + \beta\,|\,v\,\rangle) = \alpha^*\,\mathcal{T}\,|\,u\,\rangle + \beta^*\,\mathcal{T}\,|\,v\,\rangle
$$

$$
\langle\,u\,|\,\mathcal{T}\,\mathcal{T}^{\dagger}\,|\,v\,\rangle = \langle\,v\,|\,u\,\rangle
$$

and of the definitive transformation rules

$$
\mathcal{T}\,\mathbf{P}\,\mathcal{T}^{-1} = -\,\mathbf{P} \qquad\qquad \mathcal{T}\,L_{\mu\nu}\,\mathcal{T}^{-1} = -\,L_{\mu\nu}
$$

$$
\mathcal{T}\,a_{\mathbf{k}}\,\mathcal{T}^{-1} = a_{-\,\mathbf{k}} \qquad\qquad [\,\mathcal{T}\,,\,P_0\,] = 0
$$

2. To be definite, consider Graßmann valued Dirac spinor fields, which obey the complex conjugation rule $(\,\psi_1\psi_2\,)^* = \psi_2^*\psi_1^* = -\,\psi_1^*\psi_2^*$. Thus, in the chiral representation of the Clifford algebra - with γ^0 and γ_5 real matrices - we obtain

$$
\begin{aligned}
\overline{\psi^{\,c}}\,i\partial\!\!\!/\,\psi^{\,c} &= \psi^{\top}\gamma^{0}\gamma^{2}\,\gamma^{\mu}\,\gamma^{2}\,i\partial_\mu\psi^{\,*} = \psi^{\top}\gamma^{0}\,\gamma^{2\,*}\,i\partial_\mu\psi^{\,*} \\
&= \left(\,\overline{\psi}\,i\partial\!\!\!/\psi\,\right)^* = \overline{\psi}\,i\partial\!\!\!/\psi \\
\overline{\psi^{\,c}}\,\psi^{\,c} &= -\,\psi^{\top}\gamma^{2}\,\gamma^{0}\,\gamma^{2}\,\psi^{\,*} = -\,\psi^{\top}\gamma^{0}\,\psi^{\,*} = (\,\overline{\psi}\,\psi\,)^* = \overline{\psi}\,\psi \\
\overline{\psi^{\,c}}\,\gamma_5\,\psi^{\,c} &= -\,\psi^{\top}\gamma^{2}\,\gamma^{0}\,\gamma_5\,\gamma^{2}\,\psi^{\,*} \\
&= \psi^{\top}\gamma^{0}\,\gamma_5\,\psi^{\,*} = -(\,\overline{\psi}\,\gamma_5\,\psi\,)^* = -\,\overline{\psi}\,\gamma_5\,\psi \\
\overline{\psi^{\,c}}\,\gamma^{\mu}\,\psi^{\,c} &= \psi^{\top}\gamma^{0}\gamma^{2}\,\gamma^{\mu}\,\gamma^{2}\,\psi^{\,*} = \psi^{\top}\gamma^{0}\,\gamma^{2\,*}\,\psi^{\,*} \\
&= -\left(\,\overline{\psi}\,\gamma^{\mu}\,\psi\,\right)^* = -\,\overline{\psi}\,\gamma^{\mu}\,\psi \\
\overline{\psi^{\,c}}\,\gamma^{\mu}\,\gamma_5\,\psi^{\,c} &= -\,\psi^{\top}\gamma^{0}\gamma^{2}\,\gamma^{\mu}\,\gamma^{2}\,\gamma_5\,\psi^{\,*} = -\,\psi^{\top}\gamma^{0}\,\gamma^{2\,*}\,\gamma_5\,\psi^{\,*} \\
&= \left(\,\overline{\psi}\,\gamma^{\mu}\,\gamma_5\,\psi\,\right)^* = \overline{\psi}\,\gamma^{\mu}\,\gamma_5\,\psi
\end{aligned}
$$

3. In order to solve Maxwell equations in terms of the gauge potential it is necessary to introduce the so called subsidiary or auxiliary condition. If Lorentz covariance has to be manifestly maintained, then it is convenient to

select the simplest choice: namely, the so called Feynman gauge. Hence we start from the classical Lagrangian for the radiation field

$$\mathcal{L} = -\,\tfrac{1}{4}\,F_{\mu\nu}\,F^{\,\mu\nu} + \tfrac{1}{2}\,B^{\,2} + A^{\mu}\partial_{\mu}B$$

where

$$F_{0k} = E^{\,k} = -\,\partial_0 A^k - \,\nabla_k A^0 \qquad\qquad F_{rs} = \varepsilon_{srk}\,B^{\,k} \qquad \varepsilon_{123} = 1$$

while $B(x)$ is the auxiliary scalar field, so that the Euler-Lagrange equations read

$$\begin{aligned} \partial^{\,\mu} F_{\mu\nu} + \partial_{\nu} B = 0 \\ \partial \cdot A = B \end{aligned}$$

and can be recast in the simplest form

$$\Box A^{\mu}(x) = 0 \qquad\qquad \partial \cdot A(x) = B(x)$$

that imply in turn $\Box B(x) = 0$. Now, in the classical case the field equations can be further simplified, without loss of generality, by setting $B(x) \equiv 0$, that actually corresponds to the Lorenz condition. In so doing we obtain the most general solution in the form of a normal modes expansion

$$A^{\mu}(x) = \int \mathrm{D}k \sum_{A=1,2,L} g_A(k)\,\varepsilon_A^{\,\mu}(k)\,e^{-i\,k\cdot x} \;+\; \text{c.c.} \qquad (\,k_0 = \mathrm{k} = |\,\mathbf{k}\,|\,)$$

with $\mathrm{D}k = \mathrm{d}\mathbf{k}/(2\pi)^3 2\mathrm{k}$ and $g_A(k)$ arbitrary complex coefficients. The three linear and real polarization vectors are defined to be

$$\varepsilon_A^{\,\mu}(k) = \left\{ \begin{array}{lll} (0, \boldsymbol{\varepsilon}_A(\mathbf{k}) & \boldsymbol{\varepsilon}_A(\mathbf{k}) \cdot \mathbf{k} = 0 & \text{for} \quad A = 1,2 \\ (\,1, \mathbf{k}/\mathrm{k}\,) & \quad k_0^2 = \mathrm{k}^2 & \text{for} \quad A = L \end{array} \right.$$

in such a manner that the Lorenz condition holds always true and $B = 0$. Notice that in such a circumstance we recover the second pair of the Maxwell equations $\partial^{\,\mu} F_{\mu\nu} = 0$ for the radiation field. In the quantum case it is utmost convenient to keep the Feynman gauge, which endorses locality of the gauge potential operator, so that we find

$$A^{\mu}(x) = \int \mathrm{D}k \sum_{A=1,2,L,S} g_A(k)\,\varepsilon_A^{\,\mu}(k)\,e^{-\,i\,k\cdot x} \;+\; \text{H.c.} \qquad (\,k_0 = \mathrm{k}\,)$$

where the scalar polarization vector $\varepsilon_S^{\,\mu} = \frac{1}{2}(\,1, -\,\mathbf{k}/\mathrm{k}\,)$ has been employed so that

$$i\partial \cdot A(x) = iB(x) = \int \mathrm{D}k\; \mathrm{k}\, g_S(k)\, e^{-\,i\,k\cdot x} -\; \text{H.c.} \qquad (\,k_0 = \mathrm{k}\,)$$

In the quantum case the complex coefficients of the classical normal modes expansion will turn into creation and destruction operators which fulfill the canonical commutation relations

$$[\, g_A(k)\, ,\, g_{A'}(k')\,] = 0 \qquad\qquad [\, g_A(k)\, ,\, g^\dagger_{A'}(k')\,] = (2\pi)^3 2\mathrm{k}\, \delta(\mathbf{k} - \mathbf{k}')\, \eta_{AA'}$$

with

$$\eta_{AB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad\qquad (\, A, B = 1, 2, L, S\,)$$

The second pair of the Maxwell equations is recovered only in the physical subspace $\mathcal{H}_{\,\mathrm{phys}} \subset \mathcal{F}$ of the Fock space, which is selected by the auxiliary condition

$$B^{\,(-)}(x)|\mathrm{phys}\rangle = 0 \quad \Longleftrightarrow \quad g_S(k)|\mathrm{phys}\rangle = 0 \quad \forall\, \mathbf{k} \in \mathbb{R}^3$$

that yields

$$\langle \mathrm{phys}|\, \partial^{\,\mu} F_{\mu\nu} + \partial_\nu B\, |\mathrm{phys}'\rangle = \langle \mathrm{phys}|\, \partial^{\,\mu} F_{\mu\nu}\, |\mathrm{phys}'\rangle = 0$$

From the canonical commutation relations for the gauge potential in the Feynman gauge, *viz.*,

$$[\, A^\mu(x)\, ,\, A^\nu(0)\,] = i\hbar\, g^{\,\mu\nu}\, D_0(x)$$

where $D_0(x) = \lim_{m\to 0} D(x; m)$ is the mass-less Pauli-Jordan distribution, we readily obtain the gauge and translation invariant commutation relations

$$\begin{aligned} [\, F_{\mu\nu}(x)\, ,\, F_{\rho\sigma}(0)\,] \;&=\; (-i\hbar)\{g_{\nu\sigma}\, \partial_\mu \partial_\rho - g_{\mu\sigma}\, \partial_\nu \partial_\rho \\ &-\; g_{\nu\rho}\, \partial_\mu \partial_\sigma + g_{\mu\rho}\, \partial_\nu \partial_\sigma\} D_0(x) \end{aligned}$$

whence

$$[\, B_x(\mathrm{x})\, , B_x(0)\,] = i\hbar\, \big(\, \triangle - \partial_x^2 \big)\, D_0(\mathrm{x})$$
$$[\, B_x(\mathrm{x})\, , B_y(0)\,] = -\, i\hbar\, \partial_x \partial_y\, D_0(\mathrm{x})$$

et cetera, where $\mathrm{x} = (t, x, y, z)$ and $(B_x, B_y, B_z) = (F_{32}, F_{13}, F_{21})$ as usual.

FIELD THEORY **2.**

The Lagrangian is invariant under the full Lorentz group and the discrete charge conjugation symmetry, under the internal SU(2) Isospin transforms on the spinor fields

$$\Psi(x) \quad \longmapsto \quad \Psi'(x) = \exp\big\{\tfrac{1}{2}\, i\sigma_a\, \theta_a\big\}\, \Psi(x) \qquad\qquad (\, a = 1, 2, 3\,)$$

as well as the overall phase transformation on the SU(2) spinor doublet

$$\Psi(x) \quad \longmapsto \quad \Psi'(x) = \mathrm{e}^{i\varphi}\,\Psi(x)$$

where σ_a are the Pauli matrices while

$$0 \le \theta < 2\pi \qquad 0 \le \varphi < 2\pi \qquad \theta = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2}$$

are the canonical coordinates of the internal symmetry group SU(2)×U(1). The invariance under the Abelian group of the phase transformations leads to conservation of the barion number B. Thus, if we measure the charge Q in units of the proton charge e, then we can write the relation $Q = T_3 + \frac{1}{2}B$ where

$$\begin{aligned}
T_3 &= \int \mathrm{d}\mathbf{x}\,\Psi^{\dagger}(t,\mathbf{x})\,\tfrac{1}{2}\,\sigma_3\Psi(t,\mathbf{x}) \\
&= \tfrac{1}{2}\int \mathrm{d}\mathbf{x}\,[\,p^{\dagger}(t,\mathbf{x})\,p(t,\mathbf{x}) - n^{\dagger}(t,\mathbf{x})\,n(t,\mathbf{x})\,] \\
B &= \int \mathrm{d}\mathbf{x}\,[\,p^{\dagger}(t,\mathbf{x})\,p(t,\mathbf{x}) + n^{\dagger}(t,\mathbf{x})\,n(t,\mathbf{x})\,] \\
\frac{Q}{e} &= \int \mathrm{d}\mathbf{x}\;p^{\dagger}(t,\mathbf{x})\,p(t,\mathbf{x})
\end{aligned}$$

The momentum space Feynman rules are the very same for both kinds of Nucleons as well as the Feynman rules for the incoming and outgoing particles and antiparticles: namely

- pion propagator: $D_F(k) = i\,[\,k^2 - m^2 + i\varepsilon\,]^{-1}$
- spinor propagator: $S^F_{\alpha\beta}(p) = i(\not p + M)_{\alpha\beta}\,(p^2 - M^2 + i\varepsilon)^{-1}$
- pion-Nucleon-Nucleon vertex: $-\,ig \qquad (\,p_1 + k - p_2 = 0\,)$
- for each loop of internal line labeled by ℓ : $\int \mathrm{d}^4\ell/(2\pi)^4$
- a factor $(-\,1)$ for each fermion loop
- incoming Nucleon: $u_r(p)$
- outgoing Nucleon: $\bar{u}_{r'}(p')$
- incoming anti-Nucleon: $\bar{v}_r(p)$
- outgoing anti-Nucleon: $v_{r'}(p')$

The above Feynman rules give at once the lowest order $O(g^2)$ amplitude for the $\pi^0 N$ elastic scattering: namely,

$$
\begin{aligned}
& i\,\mathcal{M}_{rr'}(\mathbf{k},\mathbf{k}') = \bar{u}_{r'}(p')\,(-ig)\,S(p+k)\,(-ig)\,u_r(p) + \{k \leftrightarrow -k'\} \\
= & \;-ig^2\,\bar{u}_{r'}(p')\left[\frac{\not p + \not k + M}{(p+k)^2 - M^2} + \frac{\not p - \not k' + M}{(p-k')^2 - M^2}\right] u_r(p) \\
= & \;-ig^2\,\bar{u}_{r'}(p')\left[\frac{\not k + 2M}{(p+k)^2 - M^2} - \frac{\not k' - 2M}{(p-k')^2 - M^2}\right] u_r(p)
\end{aligned}
$$

where

$$
p + k - k' = p' \qquad k^2 = k'^2 = m^2 \qquad p^2 = p'^2 = M^2
$$

whereas use has been made of the spin-states equation $(\not p - M)\,u_r(p) = 0$. Moreover, if we select the incoming Nucleon rest frame $\mathbf{p} = 0$ we find

$$
(p+k)^2 - M^2 = m^2 + 2M\omega \qquad (p-k')^2 - M^2 = m^2 - 2M\omega'
$$

where $k^\mu = (\omega, \mathbf{k})$ with $\omega \approx |\mathbf{k}|$, whereas $k'_\mu = (\omega', -\mathbf{k}')$ with $\omega' \approx |\mathbf{k}'|$, in such a manner that we can write

$$
\begin{aligned}
& \mathcal{M}_{rr'}(\mathbf{k},\mathbf{k}') = ig^2\,\bar{u}_{r'}(p')\left[\frac{\not k + 2M}{2M\omega + m^2} + \frac{\not k' - 2M}{2M\omega' - m^2}\right] u_r(p) \\
\approx & \;\frac{ig^2}{2M\omega\omega'}\,\bar{u}_{r'}(p')\,\left[2M\,(\omega' - \omega) + \not k\omega' + \not k'\omega\right] u_r(p)
\end{aligned}
$$

for ultra-relativistic incident and scattered pions. Then we can approximate

$$
\begin{aligned}
& \mathcal{M}^*_{rr'}(\mathbf{k},\mathbf{k}') \\
\approx & \;\frac{-ig^2}{2M\omega\omega'}\,\bar{u}_r(p)\left[2M\,(\omega' - \omega) + \not k\omega' + \not k'\omega\right] u_{r'}(p')
\end{aligned}
$$

and if we set

$$
Q \equiv 2M\Delta\omega + \not k\omega' + \not k'\omega \qquad (\Delta\omega \equiv \omega' - \omega)
$$

then we can definitely write

$$
\begin{aligned}
& \left\langle |\mathcal{M}(\mathbf{k},\mathbf{k}')|^2 \right\rangle = \tfrac{1}{2}\sum_{r=1,2}\sum_{r'=1,2} |\mathcal{M}^*_{rr'}(\mathbf{k},\mathbf{k}')|^2 \\
\approx & \;\frac{g^4}{8M^2\omega^2\omega'^2}\,\mathrm{tr}\left[(\not p' + M)\,Q\,(\not p + M)\,Q\right] \\
= & \;\frac{g^4}{8M^2\omega^2\omega'^2}\,\mathrm{tr}\left[(\not p + M + \not k - \not k')\,Q\,(\not p + M)\,Q\right] \\
= & \;\frac{g^4}{8M^2\omega^2\omega'^2}\,\mathrm{tr}\left[\mathbb{A}_1 + \mathbb{A}_2 + \mathbb{A}_3\right]
\end{aligned}
$$

with

$$\mathbb{A}_1 = (\not{p} + M)\, Q\, (\not{p} + M)\, Q \qquad \mathbb{A}_2 = (\not{k} - \not{k}')\, Q\, \not{p}\, Q \qquad \mathbb{A}_3 = M(\not{k} - \not{k}')\, Q^2$$

Explicit trace calculations yields

$$\begin{aligned}
\mathrm{tr}\,\mathbb{A}_1 &= \mathrm{tr}\,[\,(\not{p} + M)\, Q\, (\not{p} + M)\, Q\,] \\
&= \mathrm{tr}\,[\,\not{p}\, Q\, \not{p}\, Q\,] + 2M\,\mathrm{tr}\,[\,Q\,\not{p}\,Q\,] + M^2\,\mathrm{tr}\,Q^2 \\
&= 16M^4\Delta\omega^2 + \mathrm{tr}\,[\,\not{p}(\not{k}\omega' + \not{k}'\omega)\not{p}(\not{k}\omega' + \not{k}'\omega)\,] \\
\mathrm{tr}\,[\,\not{p}\, Q\, \not{p}\, Q\,] &\approx 16M^4\Delta\omega^2 + 8\,(\,p\cdot k\,)^2\,\omega'^2 + 8\,(\,p\cdot k'\,)^2\,\omega^2 \\
&- 8M^2\,\omega\,\omega'\,(\,k\cdot k'\,) + 16\,\omega\,\omega'\,(\,p\cdot k\,)(\,p\cdot k'\,) \\
&= 16M^4\Delta\omega^2 + 32M^2\omega^2\,\omega'^2 - 8M^2\,\omega\,\omega'\,(\,k\cdot k'\,) \\
M^2\,\mathrm{tr}\,Q^2 &\approx 16M^4\Delta\omega^2 + 8M^2\,\omega\,\omega'(\,k\cdot k'\,) \\
2M\,\mathrm{tr}\,[\,Q\,\not{p}\,Q\,] &= 8M^2\,\Delta\omega\,\mathrm{tr}\,[\,\not{p}(\not{k}\omega' + \not{k}'\omega)\,] \approx 64M^3\,\omega\,\omega'\,\Delta\omega
\end{aligned}$$

Then we definitely find

$$\begin{aligned}
\mathrm{tr}\,\mathbb{A}_1 &\approx 32M^2\,(\omega\,\omega' + M\,\Delta\omega)^2 \\
\mathrm{tr}\,\mathbb{A}_2 &= \mathrm{tr}\,[\,(\not{k} - \not{k}')\,Q\,\not{p}\,Q\,] \;=\; 4M^2\,\Delta\omega^2\,\mathrm{tr}\,[\,(\not{k} - \not{k}')\,\not{p}\,] \\
&+ \mathrm{tr}\,[\,(\not{k} - \not{k}')\,(\,\not{k}\,\omega' + \not{k}'\,\omega\,)\,\not{p}\,(\,\not{k}\,\omega' + \not{k}'\,\omega\,)\,] \\
&\approx -\,16\,M^3\,\Delta\omega^3 - 8M\omega\omega'\Delta\omega\,(\,k\cdot k'\,) \\
\mathrm{tr}\,\mathbb{A}_3 &= 4M^2\Delta\omega\;\mathrm{tr}\,[\,(\,\not{k} - \not{k}'\,)(\,\not{k}\omega' + \not{k}'\omega\,)\,] \\
&\approx -\,16M^2\Delta\omega^2\,(\,k\cdot k'\,)
\end{aligned}$$

where we have taken into account the kinematics of the initial Nucleon rest frame $\mathbf{p} = 0$ that yields

$$p\cdot k = M\omega \qquad p\cdot k' = M\omega' \qquad k\cdot k' \approx 2\omega\omega'\sin^2\frac{\theta}{2}$$

Putting altogether we eventually get

$$\begin{aligned}
\left\langle\,|\,\mathcal{M}(\mathbf{k},\mathbf{k}'\,)\,|^2\right\rangle &= 2g^4\;\mathrm{tr}\,[\,\mathbb{A}_1 + \mathbb{A}_2 + \mathbb{A}_3\,]\,(\,4M\omega\omega'\,)^{-2} \\
&\approx 2g^4\left[\,2\left(1 + \frac{M}{\omega} - \frac{M}{\omega'}\right)^2 - M\Delta\omega\left(\frac{1}{\omega} - \frac{1}{\omega'}\right)^2\right. \\
&- \left.\frac{\Delta\omega}{M}\,\sin^2\frac{\theta}{2} + 2\left(2 - \frac{\omega}{\omega'} - \frac{\omega'}{\omega}\right)\sin^2\frac{\theta}{2}\,\right]
\end{aligned}$$

Let us close with the calculation of the incident flux factor and the final phase space volume in the massless pion limit: in this limit one immediately recovers the corresponding quantities of the Compton effect, *viz.*

$$
\begin{aligned}
\mathrm{d}\sigma &= \tfrac{1}{4}\,(\,p\cdot k\,)^{-1}\;\cdot\;\tfrac{1}{2}\sum_{r,r\,'=1,2}\,|\,\mathcal{M}_{\,rr\,'}(\mathbf{k},\mathbf{k}'\,)\,|^{2}\\
&\times\;\int\frac{\mathrm{d}\mathbf{k}'}{(2\pi)^{3}\,2\omega'}\int\frac{\mathrm{d}\mathbf{p}'}{(2\pi)^{3}\,2E'}\;(2\pi)^{4}\;\delta^{(4)}(k+p-k'-p'\,)\\
&= (\,4M\omega\,)^{-1}\;\big\langle\,|\,\mathcal{M}(\mathbf{k},\mathbf{k}'\,)\,|^{2}\big\rangle\\
&\times\;\int_{0}^{\infty}\frac{\omega'\,\mathrm{d}\omega'}{(2\pi)^{3}\,2}\int\frac{\mathrm{d}\Omega}{2E'(\omega'\,)}\;(2\pi)\,\delta\,(\omega'+E'(\omega'\,)-M-\omega)
\end{aligned}
$$

in which

$$
E'(\omega'\,)\equiv\sqrt{\omega'^{\,2}+\omega^{\,2}-2\omega\omega'\,\cos\theta+M^{\,2}}
$$

$$
\mathrm{d}\Omega=\mathrm{d}\phi\,(-\,\mathrm{d}\cos\theta)\qquad\quad(\,0\le\theta\le\pi\,,\;0\le\phi\le 2\pi\,)
$$

From the theory of the tempered distributions we get the well known relation

$$
\begin{aligned}
&\int_{0}^{\infty}\frac{\omega'\,\mathrm{d}\omega'}{E'(\omega'\,)}\;\delta\,(\omega'+E'(\omega'\,)-M-\omega)\;f(\omega'\,)\\
=\;&\left[\frac{\omega'\,f(\omega'\,)}{|\,E'(\omega'\,)+\,\omega'-\omega\cos\theta\,|}\right]_{\omega'=\widetilde{\omega}'}\qquad\quad[\,\forall\,f\in\mathcal{S}(\mathbb{R})\,]
\end{aligned}
$$

where

$$
\widetilde{\omega}'+E'(\widetilde{\omega}'\,)=\omega+M\;\Leftrightarrow\;\widetilde{\omega}'\;\equiv\;\frac{\omega\,M}{M+\omega\,(1-\cos\theta)}
$$

in such a manner that

$$
\left[\frac{\omega'}{|\,E'(\omega'\,)+\,\omega'-\omega\cos\theta\,|}\right]_{\omega'=\widetilde{\omega}'}=\frac{\omega'}{M+\omega\,(1-\cos\theta)}=\frac{\widetilde{\omega}'^{\,2}}{\omega\,M}
$$

so that we come to the differential cross-section in the Compton laboratory frame $\mathbf{p}=0$: namely,

$$
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)=\frac{1}{64\pi^{2}}\cdot\frac{\omega'}{\omega}\cdot\frac{\langle\,|\,\mathcal{M}(\mathbf{k},\mathbf{k}'\,)\,|^{2}\rangle}{M^{\,2}+2M\omega\sin^{2}(\theta/2)}
$$